Mockup of Partial Examination 1.1

Luciano Battaia

October 8, 2016

Exercise 1.

a) Given

$$g(x)=\frac{e^{2x}-1}{x},$$

compute

$$\lim_{x\to 0} g(x) \quad \textit{and} \quad \lim_{x\to 0} g'(x).$$

b) If a and b are real parameters, define

$$f(x)=\begin{cases} g(x), & \text{if } x<0;\\ a(e^{-x}-1)+b, & \text{if } x\ge 0.\end{cases}$$

Say for what values of a and b the function is continuous on $\mathbb{R}$.

c) Say for what values of a and b the function is differentiable on $\mathbb{R}$.

Exercise 2. *Given the function*

$$f(x)=x-\ln(1+x),$$

a) Compute the natural domain of f.

b) Say where f is increasing or decreasing.

c) Calculate $\lim_{x\to -1^+} f(x)$.

d) Calculate

$$\lim_{x\to +\infty} \frac{\ln(1+x)}{x}.$$

e) Observe that

$$f(x)=x\left(1-\frac{\ln(1+x)}{x}\right)$$

and calculate

$$\lim_{x\to +\infty} f(x).$$

f) Say if f has a maximum and/or minimum value.

Exercise 3.

a) Compute by parts

$$\int \left(x\mathrm{e}^x\right)\mathrm{d}x.$$

b) Given

$$f(x)=\begin{cases} x^2, & \textit{if } x<0; \\ x\mathrm{e}^x, & \textit{if } x\geq 0; \end{cases},$$

compute

$$\int_{-1}^{1} f(x)\,\mathrm{d}x.$$

Solutions follow in the next three pages.

But, before having a look at the solutions,
try to solve the exercises by yourselves!!

Solution of Exercise 1

a) As we obtain the indeterminate form $0/0$, let's apply l'Hôpital's rule:

$$\lim_{x\to 0}\frac{e^{2x}-1}{x}\overset{(H)}{=}\lim_{x\to 0}\frac{2e^{2x}}{1}=2.$$

Next we compute the derivative of g:

$$g'(x)=\frac{2e^{2x}\,x-(e^{2x}-1)\,1}{x^2}=\frac{2xe^{2x}-e^{2x}+1}{x^2}.$$

If we try to compute $\lim\limits_{x\to 0} g'(x)$ we obtain again the indeterminate form $0/0$, so we apply l'Hôpital's rule:

$$\lim_{x\to 0}\frac{2xe^{2x}-e^{2x}+1}{x^2}=\lim_{x\to 0}\frac{2e^{2x}+4xe^{2x}-2e^{2x}}{2x}=\lim_{x\to 0}\frac{2e^{2x}}{1}=2.$$

b) The only problem as regards continuity may arise at 0. In order that the function is conitnuous at 0 we require

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0),$$

that is

$$\lim_{x\to 0^-}\frac{e^{2x}-1}{x}=\lim_{x\to 0^+} a(e^{-x}-1)+b=f(0)\quad\Rightarrow\quad b=2.$$

The function is continuous for $b=2$ and any choice of a.

c) The derivative of g as $x\neq 0$ is

$$f'(x)=\begin{cases} g'(x), & \text{if } x<0;\\ -ae^{-x}, & \text{if } x>0.\end{cases}$$

The function is differentiable at 0 if

$$\lim_{x\to 0^-} f'(x)=\lim_{x\to 0^+} f'(x)$$

and both limits are finite. As

$$\lim_{x\to 0^-} f'(x)=\lim_{x\to 0^-} g'(x)=2 \quad\text{and}\quad \lim_{x\to 0^-} f'(x)=\lim_{x\to 0^-}\left(-ae^{-x}\right)=-a,$$

the function is differentiable if $a=-2$ (and $b=2$ so that it is continuous).

Solution of Exercise 2

a) The only condition for the natural domain is $1+x>0$, so that the domain is $]-1,+\infty[$.
b) The derivative of the function is

$$f'(x)=1-\frac{1}{1+x}=\frac{x}{1+x}$$

hence the function is increasing if $x>0$, decreasing if $-1<x<0$. It has a relative minimum at $x=0$.
c) We have

$$\lim_{x\to-1^+} x-\ln(1+x)=-1-\ln(0^+)=1-(-\infty)=1+\infty=+\infty.$$

d) The required limit is in the indeterminate form ∞/∞, so we can apply l'Hôpital's rule:

$$\lim_{x\to+\infty}\frac{\ln(1+x)}{x}\overset{(\cancel{H})}{=}\lim_{x\to+\infty}\frac{\frac{1}{1+x}}{1}=0.$$

e) As a consequence of the last limit, we have

$$\lim_{x\to+\infty} f(x)=+\infty(1-0)=+\infty.$$

f) As a consequence of all previous calculations f han non maximum value and a minimum value of 0 attained at $x=0$.

Solution of Exercise 3

a) We have

$$\int \left(x\mathrm{e}^x\right)\mathrm{d}x = \int \left(\mathrm{e}^x\, x\right)\mathrm{d}x = \mathrm{e}^x\, x - \int \left(\mathrm{e}^x \cdot 1\right)\mathrm{d}x = x\mathrm{e}^x - \mathrm{e}^x.$$

b) In order to calculate the requested integral we must split the interval $[-1,1]$ into two parts: $[-1,0[\cup[0,1]$. We obtain

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \int_{-1}^{0} x^2\,\mathrm{d}x + \int_{0}^{1} x\mathrm{e}^x\,\mathrm{d}x = \left[\frac{x^3}{3}\right]_{-1}^{0} + \left[x\mathrm{e}^x - \mathrm{e}^x\right]_{0}^{1} =$$

$$= \left[0 - \left(\frac{-1}{3}\right)\right] + \left[(\mathrm{e}-\mathrm{e}) - (0-1)\right] = \frac{1}{3} + 1 = \frac{4}{3}.$$