Università Ca' Foscari di Venezia - Dipartimento di Economia - A.A.2016-2017

Mathematics (Curriculum Economics, Markets and Finance)

Mockup of Partial Examination - 1.2

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October 8, 2016

Exercise 1. *Given the function*

$$f(x) = e^{\frac{2x}{x^2+3}},$$

a) compute the natural domain and the limits at the boundary of the domain;
b) say where f is increasing or decreasing;
c) compute, if they exist, the maximum and minimum of f.

Exercise 2. *Consider the function*

$$f(x) = x^3 - x^2 - \frac{x^4}{4}.$$

a) Say where f is positive or negative in its natural domain.
b) Compute $f'(x)$. Say where f is increasing and decreasing. Determine all local maximum and minimum points. Determine, if they exist, the maximum and minimum value of the function.
c) Compute $f''(x)$. Say where f is convex or concave.
d) Consider f only in the interval $[0,3]$. Find the maximum and minimum values of f.

Exercise 3. *Given the function*

$$g(x) = \begin{cases} x, & \text{if } 1 \le x \le 2; \\ x+1, & \text{if } 2 < x \le 5. \end{cases},$$

a) compute a formula for the function

$$f(x) = \int_1^x g(t)\,dt;$$

b) say whether f is continuous in $[1,5]$;
c) say whether f is differentiable in $[1,5]$.

Solutions follow in the next three pages.

But, before having a look at the solutions, try to solve the exercises by yourselves!!

Solution of Exercise 1

a) The natural domain is $\mathbb{R}$. To compute the requested limits we can compute at first the limits of the exponent. By using l'Hôpital's rule or the comparison between infinites we obtain easily that

$$\lim_{x\to\pm\infty} \frac{2x}{x^2+3} = 0,$$

so we have

$$\lim_{x\to\pm\infty} f(x) = \mathrm{e}^0 = 1$$

and the line $y = 1$ is an horizontal asymptote.

b) We have

$$f'(x) = \mathrm{e}^{\frac{2x}{x^2+3}} \left(\frac{2x}{x^2+3}\right)' = \mathrm{e}^{\frac{2x}{x^2+3}} \frac{2(x^2+3)-2x(2x)}{(x^2+3)^2} = \mathrm{e}^{\frac{2x}{x^2+3}} \frac{6-2x^2}{(x^2+3)^2}.$$

The function is increasing in $]-\sqrt{3}, \sqrt{3}[$, and decreasing in $]-\infty, -\sqrt{3}[$ and in $]\sqrt{3}, +\infty[$.

c) The minimum and maximum are attained at $x = -\sqrt{3}$ and $x = \sqrt{3}$, respectively and their values are

$$\mathrm{e}^{-\sqrt{3}/3} \quad \text{and} \quad \mathrm{e}^{\sqrt{3}/3}.$$

Solution of Exercise 2

a) We have

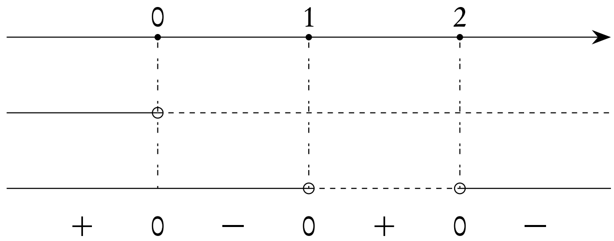
$$f(x) = -\frac{x^2}{4}\left(x^2 - 4x + 4\right) = -\frac{x^2}{4}(x-2)^2.$$

The function is never positive and is zero when $x = 0$ or $x = 2$.

b) We have

$$f'(x) = 3x^2 - 2x - x^3 = -x(x^2 - 3x + 2).$$

We can now find where f' is positive or negative, using the traditional graph of signs.



The function is increasing in $]-\infty, 0[$ and $]1, 2[$, decreasing in $]0, 1[$ and $]2, +\infty[$. There are two (local and global) maximum points, $x = 0$ and $x = 2$, a local minimum point, $x = 1$. There is no global minimum value, while the global maximum value is 0.

c) We have

$$f''(x) = -3x^2 + 6x - 2.$$

This second derivative is positive in

$$\left]1 - \frac{\sqrt{3}}{3}, 1 + \frac{\sqrt{3}}{3}\right[$$

and in this interval the function is convex. Outside it is concave.

d) If the function is restricted in the interval $[0, 3]$, the maximum and minimum exist and, as a consequence of previous calculations, we conclude that the maximum is 0 and the minimum $f(3) = -9/4$.

Solution of Exercise 3

a) If $x \leq 2$ we have

$$\int_1^x g(t)\,\mathrm{d}t = \int_1^x t\,\mathrm{d}t = \left[\frac{t^2}{2}\right]_1^x = \frac{x^2}{2} - \frac{1}{2}.$$

If $x > 2$ we have

$$int_1^x g(t)\,\mathrm{d}t = \int_1^2 t\,\mathrm{d}t + int_2^x (t+1)\,\mathrm{d}t = \left[\frac{t^2}{2}\right]_1^2 + \left[\frac{t^2}{2} + t\right]_2^x = \frac{x^2}{2} + x - \frac{5}{2}.$$

In summary

$$f(x) = \begin{cases} \dfrac{x^2}{2} - \dfrac{1}{2}, & \text{if } 1 \leq x \leq 2; \\ \dfrac{x^2}{2} + x - \dfrac{5}{2}, & \text{if } 2 < x \leq 5. \end{cases}$$

b) It is easy to see that

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) = \frac{3}{2},$$

so f is continuous everywhere.

c) We have

$$f'(x) = \begin{cases} x, & \text{if } 1 \leq x < 2; \\ x+1, & \text{if } 2 < x \leq 5. \end{cases} .$$

As

$$\lim_{x \to 2^-} f'(x) = 2 \neq \lim_{x \to 2^+} f'(x) = 3,$$

the function is not differentiable at 2.