Università Ca' Foscari di Venezia - Dipartimento di Economia - A.A.2016-2017
Mathematics -1 (Curriculum Economics, Markets and Finance)

Partial Examination - A - Prof. Luciano Battaia
2016/03/11

Schematic solution of exercises

Code A

Exercise 1. *Given the function*

$$f(x)=\begin{cases} e^x-a, & \text{if } x<0 \\ -2x^2+x+2, & \text{if } 0\le x\le 1 \\ \ln(2x-1)+b, & \text{if } x>1 \end{cases},$$

a) *find a and b so that that the function is continuous everywhere;*
b) *say whether the obtained function is differentiable at $x=0$;*
c) *say whether the obtained function is differentiable at $x=1$;*
d) *compute*

$$\int_{-1}^{1} f(x)\,dx.$$

Solution. The only problems for continuity are at $x=0$ and $x=1$. Computing the limits as $x\to 0$ and $x\to 1$ we obtain what follows.

$$\lim_{x\to 0^-} e^x-a=1-a, \quad \lim_{x\to 0^+} -2x^2+x+2=2 \quad \Rightarrow \quad 1-a=2 \quad \Rightarrow \quad a=-1.$$

$$\lim_{x\to 1^-} -2x^2+x+2=1, \quad \lim_{x\to 1^+} \ln(2x-1)+b=b \quad \Rightarrow \quad b=1.$$

With these conditions the function can be written as

$$f(x)=\begin{cases} e^x+1, & \text{if } x<0 \\ -2x^2+x+2, & \text{if } 0\le x\le 1 \\ \ln(2x-1)+1, & \text{if } x>1 \end{cases}.$$

Computing the first derivative for $x\neq 0$ and $x\neq 1$ we obtain

$$f'(x)=\begin{cases} e^x, & \text{if } x<0 \\ -4x+1, & \text{if } 0<x<1 \\ \dfrac{2}{2x-1}, & \text{if } x>1 \end{cases}.$$

In order to check differentiability at 0 and 1 we need the limits of this derivative.

$$\lim_{x\to 0^-} e^x=1, \quad \lim_{x\to 0^+} -4x+1=1 \quad \Rightarrow \quad \text{the function is differentiable.}$$

$$\lim_{x\to 1^-} -4x+1=-3, \quad \lim_{x\to 1^+} \frac{2}{2x-1}=2 \quad \Rightarrow \quad \text{the function is not differentiable.}$$

For the definite integral we must split the interval in two parts.

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = \int_{-1}^{0} (\mathrm{e}^x+1)\,\mathrm{d}x + \int_{-1}^{1} (-2x^2+x+2)\,\mathrm{d}x =$$

$$= \Big[\mathrm{e}^x + x\Big]_{-1}^{0} + \Big[-\frac{2x^3}{3}+\frac{x^2}{2}+2x\Big]_0^1 =$$

$$= \big[(1+0)-(\mathrm{e}^{-1}-1)\big] + \Big[\Big(-\frac{2}{3}+\frac{1}{2}+2\Big)-(0+0+0)\Big] = \dots \quad \square$$

Exercise 2. *Given the function*

$$f(x)=x^2-9,$$

a) *find its antiderivative, say $F(x)$, for which $F(0)=-2$;*
b) *compute the local maximum and minimum points of $F(x)$;*
c) *say whether F has global maximum and/or minimum;*
d) *compute the inflection points of F.*

Solution. The set of all antiderivatives of f is

$$\int f(x)\,\mathrm{d}x = \frac{x^3}{3} - 9x + c.$$

The conditions $F(0)=-2$ is satisfied if

$$\frac{0}{3} - 9\cdot 0 + c = -2 \quad \Rightarrow \quad c=-2.$$

The sought antiderivative is

$$F(x) = \frac{x^3}{3} - 9x - 2.$$

As the derivative of F is obviously f, that is $F'(x)=x^2-9$, we conclude that F' is positive if $x<-3$ or $x>3$, negative if $-3<x<3$. So -3 is a local maximum point and 3 is a local minimum point.

We have

$$\lim_{x\to\pm\infty} F(x) = \lim_{x\to\pm\infty} x^3\left(\frac{1}{3}-\frac{9}{x^2}-\frac{2}{x^3}\right) = \pm\infty\left(\frac{1}{3}-0-0\right) = \pm\infty.$$

As a consequence of these limits the function has no global maximum nor global minimum.

Next we easily obtain $F''(x)=2x$ and this second derivative[1] is negative as $x<0$, positive as $x>0$: $x=0$ is the only inflection point. $\square$

Exercise 3. *Given the function*

$$f(x) = \begin{cases} \ln(x)+2, & \textit{if } x>1 \\ ax^3-bx+2, & \textit{if } x\le 1 \end{cases},$$

[1] Remember that in order to find inflection points we need to compute the sign of the second derivative, not only the points where it is 0. For example the second derivative of $g(x)=x^4$ is $12x^2$: this is 0 as $x=0$, but this function has no inflection points, it is always convex (the second derivative is never negative!).

a) *find a and b such that f is everywhere continuous and differentiable;*
b) *find the limits*

$$\lim_{x\to+\infty} f(x) \quad , \quad \lim_{x\to-\infty} f(x);$$

c) *consider the function only in the interval* $[-1,1]$ *and find its global maximum and minimum.*

Solution. The only point where the function can be discontinuous or not differentiable is $x = 1$. In order to check continuity we compute the limits of the functions as $x \to 1$.

$$\lim_{x\to 1^-} \ln(x)+2 = 2, \quad \lim_{x\to 1^+} ax^3 - bx + 2 = a - b + 2 \quad \Rightarrow \quad 2 = a - b + 2.$$

Let's now compute the derivative for all $x \neq 1$:

$$f'(x) = \begin{cases} \dfrac{1}{x}, & \text{if } x > 1 \\ 3ax^2 - b, & \text{if } x < 1 \end{cases},$$

In order to check differentiability we compute the limits of this derivative as $x \to 1$.

$$\lim_{x\to 1^-} \frac{1}{x} = 1, \quad \lim_{x\to 1^+} 3ax^2 - b = 3a - b \quad \Rightarrow \quad 1 = 3a - b.$$

The function is continuous *and* differentiable if and only if

$$\begin{cases} 2 = a - b + 2 \\ 1 = 3a - b \end{cases} \quad \Rightarrow \quad a = b = \frac{1}{2}.$$

Next we obtain

$$\lim_{x\to+\infty} \ln(x) + 2 = +\infty + 2 = +\infty,$$

$$\lim_{x\to+\infty} \frac{x^3}{2} - \frac{x}{2} + 2 = \lim_{x\to+\infty} x^3\left(\frac{1}{2} - \frac{1}{2x^2} + \frac{2}{x^3}\right) = -\infty\left(\frac{1}{2} - 0 + 0\right) = -\infty.$$

In the interval $[-1,1]$ the function has a global maximum and minimum because it is continuous and the interval is closed and bounded. We have, in this interval,

$$F'(x) = \frac{3x^2}{2} - \frac{1}{2}.$$

This derivative is positive if $x < -\sqrt{3}/3$ or $x > \sqrt{3}/3$, while it is negative if $-\sqrt{3}/3 < x < \sqrt{3}/3$. So $x = -\sqrt{3}/3$ is a local maximum point and $x = \sqrt{3}/3$ a local minimum point. Furthermore we have $F(-1) = 2$ and $F(1) = 2$. We can conclude that $x = -\sqrt{3}/3$ is also a global maximum point and $x = \sqrt{3}/3$ a global minimum point. The values of these maximum and minimum are

$$f\left(-\frac{\sqrt{3}}{3}\right) = \ldots \quad \text{and} \quad f\left(\frac{\sqrt{3}}{3}\right) = \ldots$$

□

Code B

The exercises are almost the same as with Code A: only limited changes in some coefficients occur. The solution strategies are exactly the same. So we only give the detailed results of the calculations.

Exercise 1. *Given the function*

$$f(x)=\begin{cases} e^x+a, & \text{if } x<0 \\ -3x^2+x+2, & \text{if } 0\le x\le 1 \\ \ln(3x-2)+b, & \text{if } x>1 \end{cases},$$

a) *find a and b so that that the function is continuous everywhere;*
b) *say whether the obtained function is differentiable at $x=0$;*
c) *say whether the obtained function is differentiable at $x=1$;*
d) *compute*

$$\int_{-1}^{1} f(x)\,dx.$$

Solution. $a=-1$, $b=0$. Differentiable at $x=0$, not differentiable at $x=1$.

$$\int_{-1}^{1} f(x)\,dx=\int_{-1}^{0}(e^x-1)\,dx+\int_{0}^{1}(-3x^2+x+2)\,dx=[e^x-x]_{-1}^{0}+\left[-x^3+\frac{x^2}{2}+2x\right]_0^1=\dots \qquad \square$$

Exercise 2. *Given the function*

$$f(x)=3x^2-8,$$

a) *find its antiderivative, say $F(x)$, for which $F(0)=-3$;*
b) *compute the local maximum and minimum points of $F(x)$;*
c) *say whether F has global maximum and/or minimum;*
d) *compute the inflection points of F.*

Solution. $F(x)=x^3-8x-3$. The points

$$-\sqrt{\frac{8}{3}} \quad \text{and} \quad \sqrt{\frac{8}{3}}$$

are, respectively, a local maximum and a local minimum point. The function has no global maximum or minimum because the limits $x\to\pm\infty$ are $\pm\infty$. The point 0 is the only inflection point. $\square$

Exercise 3. *Given the function*

$$f(x)=\begin{cases} \ln(x)+1, & \text{if } x>1 \\ ax^3+bx+1, & \text{if } x\le 1 \end{cases},$$

a) *find a and b such that f is everywhere continuous and differentiable;*
b) *find the limits*

$$\lim_{x\to+\infty} f(x) \quad , \quad \lim_{x\to-\infty} f(x);$$

c) *consider the function only in the interval $[-1,1]$ and find its global maximum and minimum.*

Solution. The function is everywhere continuous and differentiable if and only if $a=1/2$ and $b=-1/2$. The requested limits are $\pm\infty$. The global maximum and minimum are respectivley

$$f\left(-\frac{\sqrt{3}}{3}\right) \quad \text{and} \quad f\left(\frac{\sqrt{3}}{3}\right). \qquad \square$$