Università Ca' Foscari di Venezia - Dipartimento di Economia - A.A.2016-2017
Mathematics (Curriculum Economics, Markets and Finance)

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Schematic solution

Exercise 1. *Consider the function* $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} a\,x\,e^x, & \textit{if} \quad x \le 0; \\ b - \ln(x+1), & \textit{if} \quad x > 0. \end{cases}$$

a) *Find* a *and* b *so that the function is continuous and differentiable everywhere.*
b) *Find the limit of* f *as* $x \to +\infty$.
c) *Observe that*

$$x e^x = \frac{x}{e^{-x}}$$

and find the limit of f *as* $x \to -\infty$.
d) *Find all local maximum and minimum points of* f.
e) *Say whether* f *has global maximum and/or minimum.*
f) *Find the inflection points of* f *in the interval* $]-\infty, 0[$.

Solution. As

$$\lim_{x\to 0^-} f(x) = 0 \quad \text{and} \quad \lim_{x\to 0^+} f(x) = b,$$

the condition for continuity is $b = 0$. Next let's derive the function

$$f'(x) = \begin{cases} a e^x + a x e^x, & \text{if} \quad x < 0; \\ -\dfrac{1}{x+1}, & \text{if} \quad x > 0. \end{cases}$$

As

$$\lim_{x\to 0^-} f'(x) = a \quad \text{and} \quad \lim_{x\to 0^+} f'(x) = -1,$$

the condition for differentiability is $a = -1$.

We next have

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} -\ln(x+1) = -\infty$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} -x e^x = \lim_{x\to-\infty} -\frac{x}{e^{-x}} \overset{(H)}{=} \lim_{x\to-\infty} \frac{1}{e^{-x}} = \left[\frac{1}{+\infty}\right] = 0.$$

The derivative of f is

$$f'(x) = \begin{cases} -e^x - x e^x = -e^x(x+1), & \text{if} \quad x \le 0; \\ -\dfrac{1}{x+1}, & \text{if} \quad x > 0. \end{cases}$$

This derivative is positive for $x < -1$, negative for $x > -1$, and we have $f'(-1) = 0$. This means we have only a local maximum at $x = -1$, and no local minimum points.

As a consequence of previous calculations and limits we can conclude that the function has a global maximum (at $x = -1$) and no global minimum.

For x<0 the second derivative is

$$f''(x) = -e^x - e^x - xe^x = -e^x(x+2).$$

This derivative is positive for $x < -2$ and negative for $-2 < x < 0$, so $x = -2$ is an inflection point. □

Exercise 2. *Consider the function* $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x^2 + x.$$

a) *Find the antiderivative $F(x)$ for which $F(1) = 1$.*
b) *Find the local maximum and minimum points of F.*
c) *Find the global maximum and minimum of F (if they exist).*
d) *Find the inflection points od F.*

Solution. With a straightforward calculation we find that

$$F(x) = \frac{x^3}{3} + \frac{x^2}{2} + c.$$

So we must have

$$F(1) = \frac{1}{3} + \frac{1}{2} + c = 1 \quad \Rightarrow \quad c = \frac{1}{6}.$$

As $F'(x) = f(x) = x^2 + x$, the derivative is positive for $x < -1$ and for $x > 0$, while it is negative in $-1 < x < 0$. This means that there is a maximum for F at $x = -1$ and a minimum at $x = 0$.

The calculation of the limit of F as $x \to +\infty$ is straightforward and its value is $+\infty$. As regards $-\infty$ we have

$$\lim_{x \to -\infty} = x^3 \left(\frac{1}{3} + \frac{1}{2x} + \frac{1}{6x^3} \right) = -\infty(1 + 0 + 0) = -\infty.$$

This means we have no global maximum or minimum.

The second derivative of F is $2x + 1$: the only inflection point is $x = -1/2$.

Important observation. This is a very simple exercise and calculations are straightforward! □

Exercise 3. *Consider the two variables real function*

$$f(x, y) = x^2 + y^2 + x^2y - 2y.$$

a) *Find all local maximum, minimum and saddle points.*
b) *Find global maximum and minimum on the constraint $x^2 + y^2 = 1$ without using Lagrangian multipliers.*

Solution. The necessary conditions are

$$\begin{cases} f'_x = 2x + 2xy = 0 \\ f'_y = 2y + x^2 - 2 = 0 \end{cases}.$$

From the first equation we find $2x(1+y) = 0$, that is $x = 0$ or $y = -1$. Substituting $x = 0$ in the second equation we find $y = 1$. Substituting $y = -1$ in the second equation we obtain $x^2 - 4 = 0$ that is $x = \pm 2$. We have three critical points: $(0, 1)$, $(-2, -1)$ and $(2, -1)$. The second derivatives of the function are

$$f''_{xx} = 2 + 2y, \quad f''_{xy} = f''_{yx} = 2x, \quad f''_{yy} = 2.$$

The Hessians are:

$$H(0,1) = \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = 8 > 0, \quad H(-2,-1) = \begin{vmatrix} 0 & -4 \\ -4 & 2 \end{vmatrix} = -16 < 0, \quad H(2,-1) = \begin{vmatrix} 0 & 4 \\ 4 & 2 \end{vmatrix} = -16 < 0.$$

As $f''_{xx}(0,1) = 4$, we conclude that $(0, 1)$ is a minimum point, the others are saddle points.

For the second part it is convenient to rewrite the constraint as $x^2 = 1 - y^2$, with $-1 \le y \le 1$. Substituting this expression of the constraint in the function f we obtain a single variable function, that we'll call g:

$$g(y) = 1 - y^2 + y^2 + (1 - y^2)y - 2y = -y^3 - y + 1.$$

The derivative of this one variable function is $g'(y) = -3y^2 - 1$ and is always negative: the function g is always decreasing. So the maximum is on the left boundary of the domain ($y = -1$)) and the minimum on the right boundary ($y = 1$). The corresponding maximum and minimum values for g (and also for f) are 3 and -1. □

Exercise 4. *Consider the linear system*

$$\begin{cases} x + 2y - z = 4 \\ 2x - y + 2z = -1 \\ 2x + z = 1 \end{cases}.$$

Prove that it is consistent and solve it, both using Cramer's rule and the inverse matrix strategy.

Solution. The augmented matrix of the system is:

$$A|b = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & -1 & 2 & -1 \\ 2 & 0 & 1 & 1 \end{array}\right).$$

As the matrix A is a 3×3 one, while $A|b$ is a 3×4 one, the rank of both must be less than or equal to 3. The determinant of A is 1. So the rank of A is 3, and, a fortiori 3 is also the rank of $A|b$: the system is consistent.

The solution by Cramer's rule is straightforward:

$$x = \frac{\begin{vmatrix} 4 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 0 & 1 \end{vmatrix}}{1} = 1, \quad y = \frac{\begin{vmatrix} 1 & 4 & -1 \\ 2 & -1 & 2 \\ 2 & 1 & 1 \end{vmatrix}}{1} = 1, \quad z = \frac{\begin{vmatrix} 1 & 2 & 4 \\ 2 & -1 & -1 \\ 2 & 0 & 1 \end{vmatrix}}{1} = -1.$$

As the determinat of A is not 0, the matrix has an inverse and we easily obtain:

$$A^{-1} = \begin{pmatrix} -1 & -2 & 3 \\ 2 & 3 & -4 \\ 2 & 4 & -5 \end{pmatrix}.$$

To solve the system using the inverse matrix strategy we must multiply the inverse by the column vector $\vec{b}$:

$$\begin{pmatrix} -1 & -2 & 3 \\ 2 & 3 & -4 \\ 2 & 4 & -5 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

exactly as before. □

Exercise 5. *Consider the vectors*

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} k \\ -1 \\ 2 \\ -k \end{pmatrix}.$$

a) *Find for which values of k they are linearly independent.*
b) *Set $k = 1$ and write $\vec{v}_4$ as a linear combination of the others.*

Solution. The 4 vectors are independent if and only if the matrix whose columns are the given vectors has rank 4. As this matrix is a 4×4 matrix we check its determinant:

$$\text{Det} \begin{pmatrix} 1 & 0 & 0 & k \\ -2 & 1 & 0 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -k \end{pmatrix} = -3 + 3k.$$

This means that the vectors are independent if and only if $k \neq 1$.

As a consequence of the previous result, if $k = 1$ the vectors are dependent:at least one is a linear combination of the others. We need to check if $\vec{v}_4$ is a linear combination of the others:

$$\vec{v}_4 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3,$$

that is

$$\begin{pmatrix} 1 \\ -1 \\ 2 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ -2c_1 + c_2 \\ c_2 - c_3 \\ c_3 \end{pmatrix}.$$

This condition can be writtem as a linear system:

$$\begin{cases} c_1 = 1 \\ -2c_1 + c_2 = -1 \\ c_2 - c_3 = 2 \\ c_3 = -1 \end{cases},$$

whose solution is immediate: $c_1 = 1$, $c_2 = 1$, $c_3 = -1$. The linear combination is

$$\vec{v}_4 = \vec{v}_1 + \vec{v}_2 - \vec{v}_3.$$ □