Università Ca' Foscari di Venezia - Dipartimento di Economia - A.A.2016-2017

Mathematics (Curriculum Economics, Markets and Finance)

A dash of derivatives

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1 Tangents to curves

The concept of tangent, at a given point P, to a circumference with center O is well known. There are three possible definitions of this concept.

1. The tangent at P is the exclusive straight line through P that has only one point in common with the circumference. All other lines through P have exactly one other point Q in common with the circumference. This is by far the best-known definition of tangent to a circumference. This situation is illustrated in figure 1.

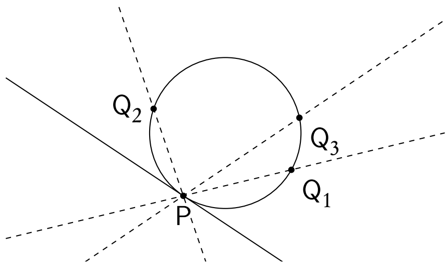

Figure 1 *The tangent as the exclusive line that has only one point in common with a circumference.*

2. The tangent at P is the straight line perpendicular to the radius $\overline{\mathsf{OP}}$. This is by far the most useful definition: if you want to effectively draw the tangent line in a geometrical drawing, the simplest way is to draw the perpendicular to the ray; this construction is also the more precise one. See figure 2
3. The third definition is of a completely different kind and involves the concept of limit. The process may be described as follows. Given the point P, take another point Q on the circumference and consider the straight line through P and Q. This line is called a *secant*. If you move Q on the circumference, the line changes, and when Q is over P the secant disappears: there infinitely many straight lines passing through one only point. Remember that when considering the limit of a function as $x \to a$, we are not interested in what happens when x is exactly a, but in what happens as x gets closer and closer to a. In a similar way we can consider the line PQ as a function of Q and the

$$\lim_{\mathsf{Q}\to\mathsf{P}} \mathsf{PQ}$$

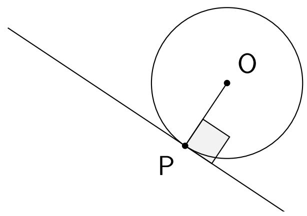

Figure 2 *The tangent as the straight line perpendicular to the radius*

that is the behaviour of the secant line as Q gets closer and closer to P.
What we are interested in is this *limit line*. It is almost obvious that this limit line is exactly the tangent obtained by the two previous definitions. In an simple and intuitive way one can say that, while the two previous definitions where "static", this is a "dynamic" definition as it involves a movement of the point Q. See figure 3

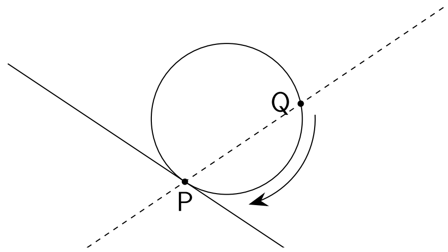

Figure 3 *The tangent as the limit position of a secant*

Let us now consider the problem of defining the tangent to a generic curve. Can we extend the previous definitions to a situation like this? Certainly a generic curve does not have a radius, so the second definition can't be used. As regards the first if we consider a parabola, see figure 4, it is immediate to note that through a given point P there are two lines that have only one point in common with the curve, but only one is what intuitively we would like to call "tangent".

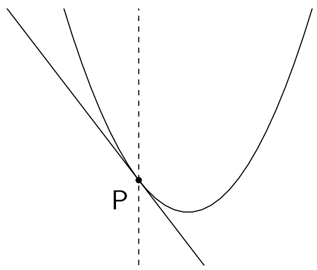

Figure 4 *Two lines passing trough a point that have only one point in common with a parabola*

But, even worse, we can consider curves where the line that we intuitively would like to call "tangent" has many points in common with a curve, at the limit also infinitely many: such a situation is illustrated in figure 5.

The only reasonable way to extend the concept of tangent to general curves is to use the third "dynamical" definition.

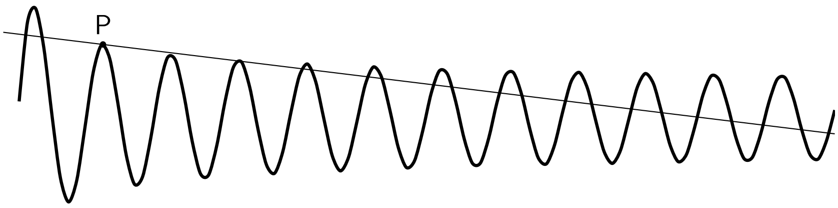

Figure 5 *A tangent line that has many points in common with a curve*

See section 6.2 of your textbook for details. In addition to the content of this section of the textbook, only note that, given a function $f(x)$, different symbols are used for derivatives. The most common ones are:

— $f'(x)$

— $\mathrm{D}\big(f(x)\big)$.

— $\dfrac{\mathrm{d}}{\mathrm{d}x}\big(f(x)\big)$.

The last symbol is particularly useful to highlight the name of the variable taken into consideration. In fact in many economic applications a function may have many variables, but sometimes we focuse attention only on one of them.

Observe that the limit involved in the definition, at least for continuous functions, is always in the form $0/0$. In fact if f is continuous the limit can be calculated by simple substitution, so we have

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\frac{f(a)-f(a)}{0}=\frac{0}{0}\,.$$

This means that the calculation of a derivative is, in principle, not easy at all. Let us consider two examples.

Example 1. *Calculate the derivative of $f(x)=x^2$ at a generic point x.*

We obtain:

$$\lim_{h\to 0}\frac{(x+h)^2-x}{h}=\lim_{h\to 0}\frac{x^2+2hx+h^2-x^2}{h}=\lim_{h\to 0}\frac{2x\cancel{h}+h^{\cancel{2}}}{\cancel{h}}=\lim_{h\to 0}2x+h=2x\,.$$

Example 2. *Calculate the derivative of $f(x)=\sqrt{x}$ at a generic point $x\neq 0$.*

We obtain:

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=$$
$$=\lim_{h\to 0}\frac{x+h-x}{h\big(\sqrt{x+h}+\sqrt{x}\big)}=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}\big(\sqrt{x+h}+\sqrt{x}\big)}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}\,.$$

When x is 0 this calculation is not correct. In this case only the limit as $h\to 0^+$ is allowed in that the natural domain of the function is $[0,+\infty[$, and we obtain:

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{\sqrt{h}}{h}=\lim_{h\to 0^+}\frac{1}{\sqrt{h}}=+\infty.$$

This last result means that the tangent to the graph of this function as $x=0$ is vertical, and in effect this is the case, as figure 6 shows.

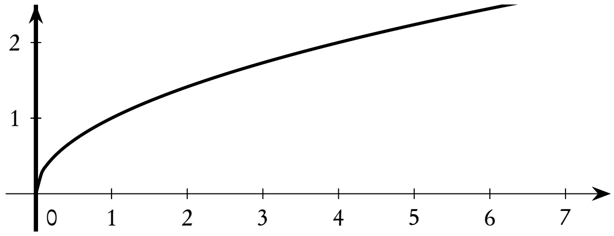

Figure 6 *The tangent to the graph of $f(x)=\sqrt{x}$, as $x=0$, is a vertical line*

2 Rules for derivatives

Despite the inherent difficulty of calculating derivatives by means of the definition, it turns out that, at least for the elementary functions that are the main topic of our course, some theorems can be established that make this calculation almost automatic. We'll briefly summarize the rules we are interested in, with some examples on how to use them.

1. Elementary functions.
   a) $f(x)=k, \quad f'(x)=0$.
   b) $f(x)=x^a, \quad f'(x)=ax^{a-1}$. In particular:
      - $f(x)=x, \quad f'(x)=1$;
      - $f(x)=x^2, \quad f'(x)=2x$;
      - $f(x)=x^{1/2}=\sqrt{x}, \quad f'(x)=\dfrac{1}{2\sqrt{x}}$;
      - $f(x)=x^{-1}=\dfrac{1}{x}, \quad f'(x)=-x^{-2}=-\dfrac{1}{x^2}$.
   c) $f(x)=\mathrm{e}^x, \quad f'(x)=\mathrm{e}^x$.
   d) $f(x)=\ln x, \quad f'(x)=\dfrac{1}{x}$.
2. Sums, products and quotients.
   a) $\big(f(x)\pm g(x)\big)'=f'(x)\pm g'(x)$.
   b) $\big(f(x)\cdot g(x)\big)'=f'(x)\cdot g(x)+f(x)\cdot g'(x)$.
   c) $\left(\dfrac{f(x)}{g(x)}\right)'=\dfrac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{\big(g(x)\big)^2}$.

Observe that 2a means that *The derivative of a sum is the sum of the derivatives*, that is the two operations of sum and differentiation can be interchanged. The same *does not* hold for products and quotients. The rule for the product of two functions is sometimes called *chain rule*.

Example 3. *Compute the derivative of*

$$f(x)=\sqrt{x}\ln x.$$

We obtain:

$$f'(x)=\big(\sqrt{x}\big)'\ln x+\sqrt{x}\big(\ln x\big)'=\frac{1}{2\sqrt{x}}\ln x+\sqrt{x}\,\frac{1}{x}.$$

Example 4. *Compute the derivative of*

$$f(x)=x^2\,\mathrm{e}^x\,\ln x.$$

In this case we have the product of three functions, but we can always group the factors, for instance in the following way:

$$f(x) = \left(x^2\,\mathrm{e}^x\right)\ln x.$$

Now we obtain

$$f'(x) = \left(x^2\,\mathrm{e}^x\right)'\ln x + \left(x^2\,\mathrm{e}^x\right)\left(\ln x\right)' = \left(\left(x^2\right)'\mathrm{e}^x + x^2\left(\mathrm{e}^x\right)'\right)\ln x + x^2\,\mathrm{e}^x\left(\ln x\right)' =$$

$$= \left(x^2\right)'\mathrm{e}^x\,\ln x + x^2\left(\mathrm{e}^x\right)'\,\ln x + x^2\,\mathrm{e}^x\left(\ln x\right)' = 2x\,\mathrm{e}^x\,\ln x + x^2\,\mathrm{e}^x\,\ln x + x^2\,\mathrm{e}^x\,\frac{1}{x}.$$

Observe that, as a consequence of this example, you can deduce that the derivative of a product of n functions is the sum of n terms each of which contains the derivative of one of the functions multiplied by the other functions unchanged.

Example 5. *Compute the derivative of*

$$f(x) = \frac{x^3}{\mathrm{e}^x}.$$

We obtain:

$$f'(x) = \frac{\left(x^3\right)'\mathrm{e}^z - x^3\left(\mathrm{e}^x\right)'}{\left(\mathrm{e}^x\right)^2} = \frac{3x^2\,\mathrm{e}^x - x^3\,\mathrm{e}^x}{\mathrm{e}^{2x}}.$$

Example 6. *Compute the derivative of*

$$f(x) = \frac{x\mathrm{e}^x}{x^2 + x}.$$

We obtain:

$$f'(x) = \frac{\left(x\mathrm{e}^x\right)'(x^2+x) - x\mathrm{e}^x\left(x^2+x\right)'}{(x^2+x)^2} = \frac{\left(1\cdot\mathrm{e}^x + x\mathrm{e}^x\right)(x^2+x) - x\mathrm{e}^x(2x+1)}{(x^2+x)^2}.$$

3 Te derivative of composite functions

With the previous rules it is not possible to calculate the derivative of composite functions, such as $f(x) = \sqrt{x^2+1}$. The rule for composite functions is somewhat difficult to learn, even if it is in effect very simple. We'll not write here the general rule: for details see your textbook. Here we will propose a simplified notation. The general idea is the following: in the rules for differentiating elementary functions, if x is replaced by something else (we'll write $*$ to represent that "something else"), just use the same rule, with $*$ in the place of x and then *multiply by* the derivative of this "something else". Also this rule is known as *chain rule*. We repeat the formulas for elementary functions with this modification.

1. $f(x) = *^a$, $\quad f'(x) = a *^{a-1}\,(*)'$. In particular:
   - $f(x) = *^2$, $\quad f'(x) = 2*\,(*)'$;
   - $f(x) = *^{1/2} = \sqrt{*}$, $\quad f'(x) = \dfrac{1}{2\sqrt{x}}\,(*)'$;
   - $f(x) = *^{-1} = \dfrac{1}{*}$, $\quad f'(x) = -*^{-2}\,(*)' = -\dfrac{1}{*^2}\,(*)'$.
2. $f(x) = \mathrm{e}^*$, $\quad f'(x) = \mathrm{e}^*\,(*)'$.
3. $f(x) = \ln *$, $\quad f'(x) = \dfrac{1}{*}\,(*)'$.

Example 7. *Compute the derivative of*

$$f(x) = \ln\left(e^x + x^2\right).$$

We obtain:

$$f'(x) = \frac{1}{e^x + x^2}\left(e^x + x^2\right)' = \frac{1}{e^x + x^2}\left(e^x + 2x\right).$$

Example 8. *Compute the derivative of*

$$f(x) = \sqrt{e^x + x^2 + \ln x}.$$

We obtain:

$$f'(x) = \frac{1}{2\sqrt{e^x + x^2 + \ln x}}\left(e^x + x^2 + \ln x\right)' = \frac{1}{2\sqrt{e^x + x^2 + \ln x}}\left(e^x + 2x + \frac{1}{x}\right).$$

Example 9. *Compute the derivative of*

$$f(x) = \sqrt{e^{x^2+\sqrt{x}}}.$$

We obtain:

$$f'(x) = \frac{1}{2\sqrt{e^{x^2+\sqrt{x}}}}\left(e^{x^2+\sqrt{x}}\right)' = \frac{1}{2\sqrt{e^{x^2+\sqrt{x}}}}e^{x^2+\sqrt{x}}\left(x^2 + \sqrt{x}\right)' = \frac{1}{2\sqrt{e^{x^2+\sqrt{x}}}}e^{x^2+\sqrt{x}}\left(2x + \frac{1}{2\sqrt{x}}\right).$$

4 Compound or piecewise defined functions

A function that is not continuous at a given point can't have a derivative at that point. We know that, as regards our course, the simplest way to construct non continuous functions is to use piecewise definitions. Problems may arise only in correspondence with the points of separation between the various "pieces" that define the function. In order to assess whether a compound function (that as already been checked for continuity) is differentiable, proceed as in the following example.

Example 10. *Determine whether the following function is differentiable or not.*

$$f(x) = \begin{cases} x^2, & \textit{if } x \le -1; \\ x + 2, & \textit{if } -1 < x \le 1; \\ -2x^2 + 5x, & \textit{if } x > 1. \end{cases}$$

It is simple enough to check the continuity of this function at $x = -1$ and $x = 1$. To check differentiability calculate the derivative of the three parts, *skipping*(1) the separation points.

$$f'(x) = \begin{cases} 2x, & \text{if } x < -1; \\ 1, & \text{if } -1 < x < 1; \\ -4x + 5, & \text{if } x > 1. \end{cases}$$

Next calculate the limits (from the left and from the right) of the three expressions obtained: these are called, respectively, *left* and *right derivatives*. If at a given point left and right derivatives are equal, the function is differentiable, otherwise it is not. If the two derivatives are finite, but not equal, the corresponding point on the graph is called a *kink*.

[1] Pay attention to this fact!

In this case we obtain:

$$\left(\lim_{x\to -1^-} f'(x) = -2\right) \neq \left(\lim_{x\to -1^+} f'(x) = 1\right), \qquad \left(\lim_{x\to 1^-} f'(x) = 1\right) = \left(\lim_{x\to 1^+} f'(x) = 1\right).$$

So this function is not differentiable at $x = -1$, while it is at $x = 1$. We can write

$$f'(x) = \begin{cases} 2x, & \text{if } x < -1; \\ 1, & \text{if } -1 < x \leq 1; \\ -4x + 5, & \text{if } x > 1. \end{cases},$$

where we have added the equals sign in correspondence with $x = 1$ (from the left or the right: it would be the same thing). By examining the plot in figure 7 one can easily understand what it means that the function is not differentiable at -1, while it is at 1: in the first case the two curves from the left and the right join, but with two different slopes (there are two different tangents, a left and a right tangent), while in the second case the joint is smooth and the two tangents from the left and from the right coincide.

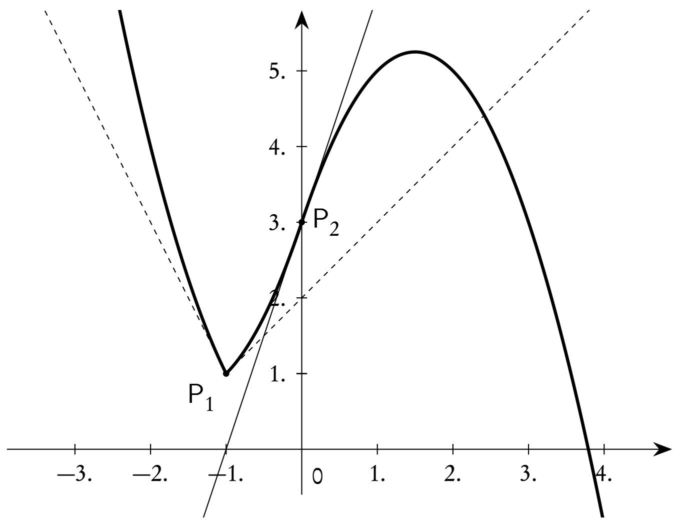

Figure 7 *A piecewise defined function everywhere continuous but not differentiable at a point*

Exercise 1. *Say for what values of a and b the following function is continuous and differentiable. Plot the corresponding graph.*

$$f(x) = \begin{cases} ax^2 + bx, & \textit{if } x < 2; \\ \sqrt{x+2}, & \textit{if } x \geq 2. \end{cases}$$

Solution. The only problem may occur at $x = 2$. Continuity is obtained if

$$\lim_{x\to 2^-} ax^2 + bx = \lim_{x\to 2^+} \sqrt{x+2} \quad \Rightarrow \quad 4a + 2b = 2.$$

Let's calculate the derivatives of the two pieces with which the function is built.

$$f'(x) = \begin{cases} 2ax + b, & \text{if } x < 2; \\ \dfrac{1}{2\sqrt{x+2}}, & \text{if } x > 2. \end{cases}$$

Differentiability is obtained if

$$\lim_{x\to 2^-} 2ax+b = \lim_{x\to 2^+} \frac{1}{2\sqrt{x+2}} \quad \Rightarrow \quad 4a+b=\frac{1}{4}\,.$$

If we now solve the system

$$\begin{cases} 4a+2b=2 \\ 4a+b=\dfrac{1}{4} \end{cases},$$

we obtain the requested values of a and b:

$$a=-\frac{3}{8},\quad b=\frac{7}{4}\,.$$

□

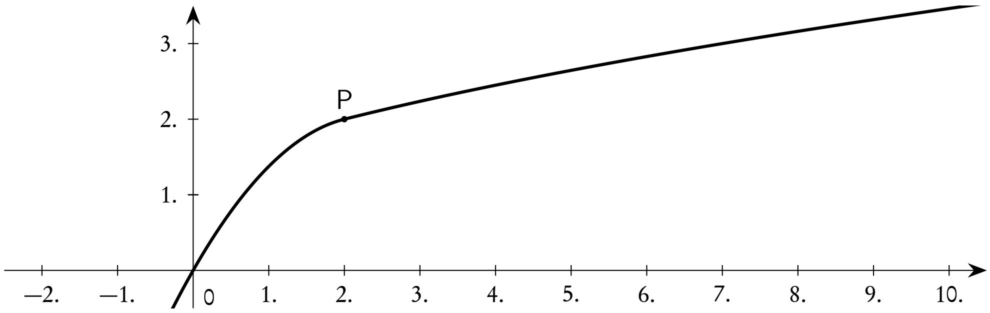

Figure 8 *The function of exercise 1*