Università Ca' Foscari di Venezia - Dipartimento di Economia - A.A.2016-2017

Mathematics (Curriculum Economics, Markets and Finance)

Global max and min for two variables functions

Luciano Battaia

September 30, 2019

1 References to the case of one variable

In the case of (twice) differentiable one variable functions the search for global maximum and minimum in a given subset of the real line is usually simple enough: you can follow the following procedure.

1. Compute the derivative and find all local maximum and minimum points, using the sign of the first derivative, or using the second derivative. Compute the values of these maxima and minima.
2. Check the behaviour of the function at the boundaries of the domain. In the cases of our interest the domain is always an interval (the real line, a half-line or a bounded interval). So you can compute the limits as x tends to the extreme points of the domain, or the value of the function at the extreme points and, by comparing these limits or these values with the previous local maxima and minima, you can easily conclude.

Let's consider some examples.

Example 1. *For the function $f(x)=x^2$ we find only one local minimum at $x=0$, with value 0. The limits as $x\to\pm\infty$ are $+\infty$, so we conclude that the function has non global maximum and has a global minimum at $x=0$, whose value is 0.*

Example 2. *For the function $f(x)=x^3-12x$ we find a local maximum at $x=-2$ and a local minimum at $x=2$, with respective values 16 and -16. The limits as $x\to\pm\infty$ are $\pm\infty$, so the function has no global maximum nor minimum.*

Example 3. *For the same function of the previous example **??**, $f(x)=x^3-12x$, but considered* only *in the interval $[-5,3]$, we find the local maximum and minimum points already considered. The values at the extremes of the domain are $f(-5)=-65$ and $f(3)=-9$, so we conclude that the function has a global minimum at $x=-5$, with value -65 and a global maximum at $x=-2$ with value 16.*

Example 4. *Again for the same function of example **??**, but now considered in the interval $[-4,5]$, we find the local maximum and minimum points already considered. The values at the extremes of the domain are $f(-4)=-16$ and $f(5)=65$, so we conclude that the function has a global minimum at $x=5$, with value 65 and a global maximum at $x=-4$ and $x=2$ with value 16.*

Example 5. *For the function $f(x)=1/x$, considered in the interval $]0,+\infty[$, we find no local maximum or minimum points. As*

$$\lim_{x\to0^+}f(x)=+\infty \quad \textit{and} \quad \lim_{x\to+\infty}f(x)=0,$$

we can conclude that the function has no global maximum or minimum. Observe that the value 0 *of the limit is never reached by the function: the graph approaches the line* $y = 0$ *(horizontal asymptote) but it never touches it.*

Example 6. *Fo the function of example ?? considered in the interval* $[1, +\infty[$*, as* $f(1) = 1$*, we conclude that there is a global maximum at* $x = 1$*, with value* 1 *and no global minimum. For the same function considered in the interval* $]0, 1]$ *we conclude that there is no global maximum, but a global minimum at* $x = 1$ *with value* 1.

Example 7. *For the function* $f(x) = 1/(x^2 + 1)$ *we find a local maximum at* $x = 0$ *with value* 1*. The limits as* $x \to \pm\infty$ *are* 0*, so we conclude that the function has a global maximum at* $x = 0$ *with value* 1*, but no global minimum.*

2 Functions of two variables

The strategy to be applied for functions of two variables is, in principle, the same, but we must keep into account the following facts.

1. The concept of increasing or decreasing function does not make sense, so we are forced to used the second derivatives test.
2. Computing limits for two variables functions is by no means easy and we have not considered this problem in our course. For this reason we'll be interested only in finding global maximum and minimum points only for functions whose domain is closed and bounded. In the case of two variables, as regards our course, the boundary of such a domain is usually a closed curve like a circumference or an ellipse, or a closed line consisting of a circumference pieces, of segments, of a parabola pieces.

This said, you can procede in a way similar to the case of one variable functions.

1. Compute the two first partial derivatives and check where they are both zero. All the points (stationary points) where these derivatives are both zero are "candidates" to be global maximum or minimum points. Usually you do not need to check whether they are maximum, minimum or saddle points (using the Hessian), unless this is specifically requested by the text of the problem.
2. Check the behaviour of the function at the boundary of the domain (a closed line as mentioned). This is usually the most difficult part of the problem and, in the cases of our interest, can be done using one of the following strategies.
   a) Use in a suitable way the equation of the boundary (or the equations of the boundary in case the are different pieces of lines) to obtain a function of one variable form the function of two variables, and then proceed as mentioned for functions of one variable.
   b) Use the Lagrangian multiplier method.

The Lagrangian multiplier method is, in principle, more general, but the first method is usually preferable, *when possible*. In this note we'll discuss only the first method using some examples.

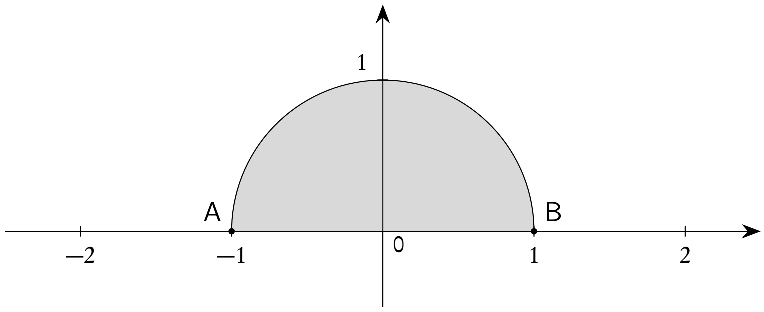
Example 8. *Find the global maximum and minimum of the function*

$$f(x, y) = x^2 - 8x + y^2 + 7$$

in the domain given by the following inequalities

$$\begin{cases} x^2 + y^2 \leq 1 \\ y \geq 0 \end{cases} .$$

Solution. Let's plot the domain.



Now we search the stationary points.

$$\begin{cases} f'_x(x,y) = 2x - 8 = 0 \\ f'_y(x,y) = 2y = 0 \end{cases}, \quad \Rightarrow \quad x = 4 \quad \text{and} \quad y = 0.$$

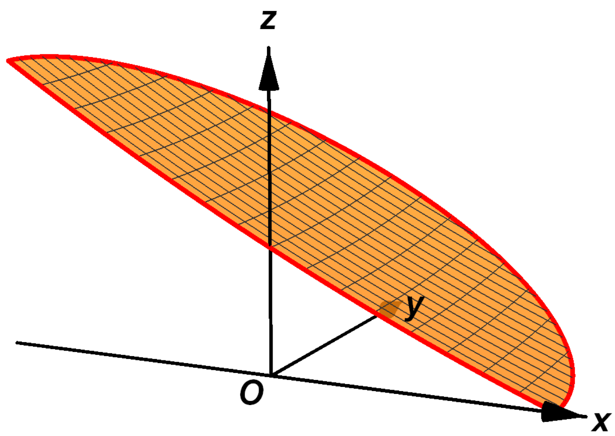
The only stationary point is $(4,0)$, that is outside our domain (so it is not important for our problem.

Now we check the boundary.

- Segment $\overline{AB}$. The equation of this segment is $y = 0$, with $-1 \le x \le 1$. By substitution into the function f we obtain the one variable function $g(x) = x^2 - 8x + 7$. This function has a global maximum at $x = -1$ and a global minimum at $x = 1$. These values correspond to the points $(-1,0)$ and $(1,0)$ of the plane. The values of the function g at -1 and 1 are, respectively, 16 and 0, and they are also the values of the function f at $(-1,0)$ and $(1,0)$.
- Arc $\overset{\frown}{AB}$. We can write the equation of this arc as $y^2 = 1 - x^2$, remembering, if needed, that $y \ge 0$. By substitution into the function f we obtain the one variable function $h(x) = -8x + 8$. This function has a global maximum at $x = -1$ and a global minimum at $x = 1$, with values 16 and 0 respectively. These points are exactly the same as those found in the segment $\overline{AB}$.

We can conclude that the function has 16 as global maximum and 0 as global minimum, in the given domain. □

Only for the sake of completeness[1], we plot the surface-graph of this function in the given domain (the units of measure are different for the three axes).



Example 9. *Find the global maximum and minimum of the function*

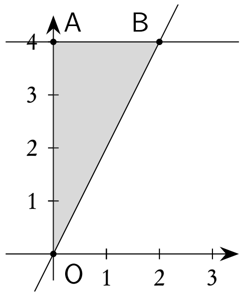
$$f(x,y) = 2x^2 - 8x + y^2 - 8y + 7$$

[1] You will not be requested to plot such a graph!

in the bounded domain limited by the lines

$$x = 0, \quad y = 4, \quad y = 2x.$$

Solution. Let's plot the domain.



Now we search the stationary points.

$$\begin{cases} f'_x(x,y) = 4x - 8 = 0 \\ f'_y(x,y) = 2y - 4 = 0 \end{cases}, \quad \Rightarrow \quad x = 2 \quad \text{and} \quad y = 4.$$

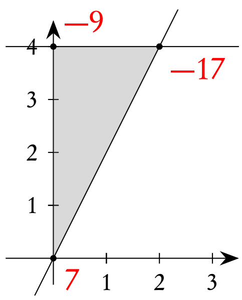
The only stationary point is $(2,4)$ (the point B) that is exactly on the boundary: it will be investigated while considering the boundary.

Now we check the boundary.

- Segment $\overline{OA}$. The equation of this segment is $x = 0$, with $0 \le y \le 4$. By substituting into the function f we obtain the one variable function $g(y) = y^2 - 8y + 7$. This function has a global maximum at $y = 0$ and a global minimum at $y = 4$, with respective values 7 and -9. These values are obviously the values of function f at $(0,0)$ and $(0,4)$.
- Segment $\overline{AB}$. The equation of this segment is $y = 4$, with $0 \le x \le 2$. The value $x = 2$ has been computed taking the intersection of the lines $y = 4$ and $y = 2x$. By substituting into the function f we obtain the one variable function $h(x) = 2x^2 - 8x - 9$. This function has a global maximum at $x = 0$ and a global minimum at $x = 2$, with respective values -9 and -17. These values are obviously the values of function f at $(0,4)$ (already calculated before) and $(2,4)$.
- Segment $\overline{OB}$. The equation of the segment is $y = 2x$, with $0 \le x \le 2$. By substituting into the function f we obtain the one variable function $p(x) = 6x^2 - 24x + 7$. This function has a global maximum at $x = 0$ and a global minimum at $x = 2$, with respective values 7 and -17. These values are obviously the values of function f at $(0,0)$ and $(2,4)$: both have been already calculated before.

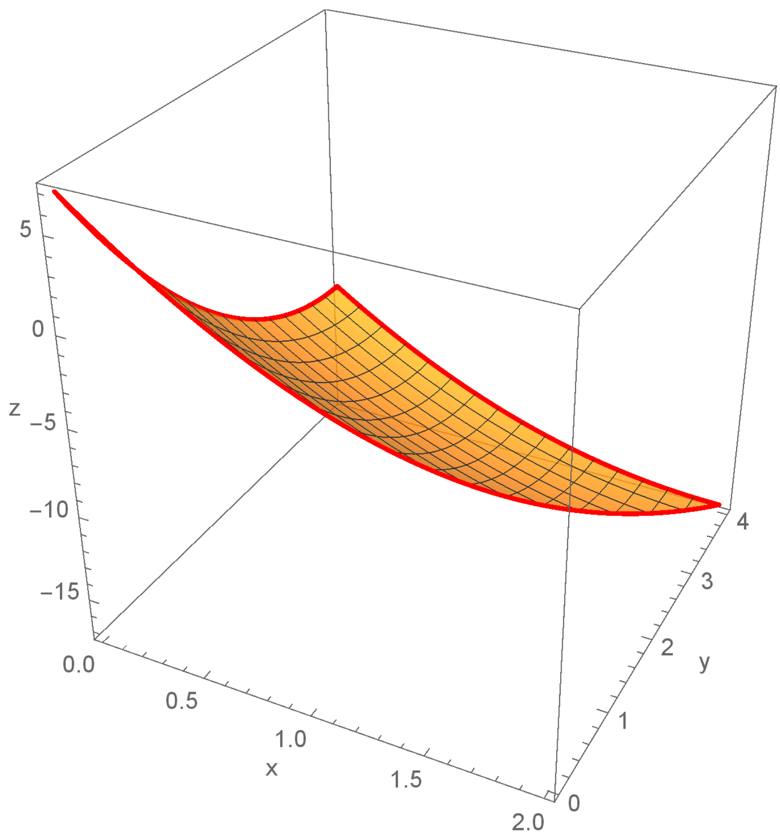
We can conclude that the function has 7 as global maximum and -17 as global minimum.

In many cases it is useful to write the values of the function f near the points of the domain were they are attained, as in the following picture.



This can help finding the correct result. □

Again only for the sake of completeness, we plot the surface-graph of this function in the given domain (the units of measure are different for the three axes).

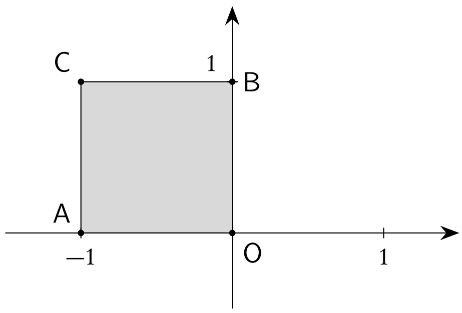


Example 10. *Find the global maximum and minimum of the function*

$$f(x,y) = x^2y - xy^2 + xy$$

in the bounded square whose vertices are $\mathsf{A} = (-1,0)$, $\mathsf{B} = (0,1)$, $\mathsf{O} = (0,0)$, *and* $\mathsf{C} = (-1,1)$.

Solution. Let's plot the domain.



Now we search the stationary points.

$$\begin{cases} f'_x(x,y) = 2xy - y^2 + y = 0 \\ f'_y(x,y) = x^2 - 2xy + x = 0 \end{cases}, \quad \Rightarrow \quad \begin{cases} y(2x - y + 1) = 0 \\ x(x - 2y + 1) = 0 \end{cases}.$$

From the first equation we obtain $y = 0$ or $2x - y + 1 = 0$, that is $y = 2x + 1$. Substituting $y = 0$ in the second equation we obtain $x = 0$ or $x = -1$. Substituting $y = 2x + 1$ in the second equation we obtain

$x = 0$ (so that $y = 1$ from the first equation) or $x = -1/3$ (so that $y = 1/3$ from the first equation). We have 4 stationary points:

$$\mathsf{A} = (-1, 0), \quad \mathsf{O} = (0, 0), \quad \mathsf{B} = (0, 1), \quad \mathsf{D} = \left(-\frac{1}{3}, \frac{1}{3}\right).$$

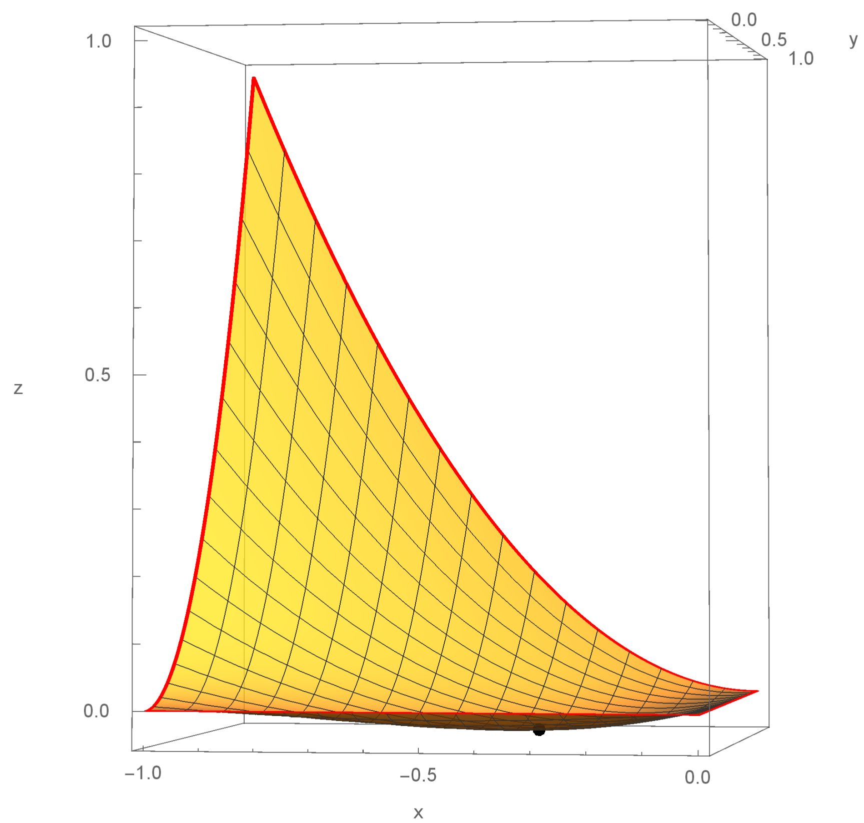
The points A, O and B are on the boundary, so they will be considered while checking the boundary. The point D is at the interior of the domain and the value of the function at this point is $-1/27$[2].

Now we check the boundary.

- Segment $\overline{\mathsf{AO}}$. The equation is $y = 0$, with $-1 \le x \le 0$. By substituting $y = 0$ into the function f we obtain a constant one variable function $g(x) = 0$. This means that in this segment the function is constantly 0.
- Segment $\overline{\mathsf{OB}}$. The equation is $x = 0$, with $0 \le y \le 1$. By substituting $x = 0$ into the function f we obtain a constant one variable function $h(y) = 0$. This means that in this segment the function is constantly 0.
- Segment $\overline{\mathsf{AC}}$. The equation is $x = -1$, with $0 \le y \le 1$. By substituting $x = -1$ into the function f we obtain the one variable function $p(y) = y^2$. This function has a global maximum at $y = 1$ and a global minimum at $y = 0$, with respective values 1 and 0.
- Segment $\overline{\mathsf{BC}}$. The equation is $y = 1$, with $-1 \le x \le 0$. By substituting $y = 1$ into the function f we obtain the one variable function $q(x) = x^2$. This function has a global maximum at $x = -1$ and a global minimum at $x = 0$, with respective values 1 and 0.

By comparing the obtained values we conclude that 1 is the global maximum of the function and $-1/27$ the global minimum, in the given domain. □

Again only for the sake of completeness, we plot the surface-graph of this function in the given domain (the units of measure are different for the three axes).



[2] It is not requested in this problem, but it is easy to verify that A, O and B are saddle points, while D is a local minimum.

A zoom of the part of this graph near the cartesian plane $z = 0$ better illustrates the situation.