A dash of integrals

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These notes are to be considered as a brief summary of the contents of chapter 9 of the textbook. Refer to this chapter for details.

Unless otherwise specified, the functions considered in these notes are continuous functions on an interval of the real line.

1 Preliminaries

The subject of integration groups together two very different problems:

1. given a function $f(x)$ try to find, if it exists, a function $F(x)$, that we call an *antiderivative* of f, such that $F'(x) = f(x)$, for all x in the domain of f;
2. given a region in the cartesian plane (usually the region between the graphs of two real valued functions), try to find its *area*.

At first glance these two problems seem to have nothing in common: however we'll see that they are strictly connected and, in particular, that the resolution of the first problem is the key to solve the second one.

Note that we used the indefinite article "an" before *antiderivative*. Indeed as the derivative of a constant is always 0, if F is an antiderivative of f, also $F + c$ is an antiderivative: if a function has an antiderivative, it has infinitely many antiderivatives.

Let's make some observation about the notations commonly used. Usually for the function f a lowercase letter is used, while a capital letter is used for one of its antiderivatives: said a little jokingly, it is as if F were the "mother" (or "father") of f! For the derivative of a function f we have introduced different notations, of which the most used are $f'(x)$ and $D\big(f(x)\big)$. For the antiderivatives the following notation, a bit more complicated, is used.

If f is a real function with domain D, the set of *all* its antiderivatives, if they exist, is represented by the symbol

$$\int f(x)\,\mathrm{d}x, \tag{1}$$

which reads *the indefinite integral of "f x d x"*.

The symbol $\int$ was first introduced by Leibniz and is based on a special letter, ſ, called "long s", standing for "summa", the latin word for "sum" or "total".

It is very important to keep in mind the fact that while the derivative of an elementary function is always an elementary function, nothing similar happens for antiderivatives: there are very simple and important elementary functions that don't have an elementary antiderivative. The most important example is the "Gauss' function" e^{-x^2}, and also the function e^{x^2}: there is no way to express

$$\int e^{\pm x^2}\,dx \tag{2}$$

in terms of elementary functions. We can say that the indefinite integral that appears in equation (2) is *impossible* in terms of elementary functions.

This is the main reason that makes the computation of antiderivatives very difficult, in general.

2 Some important rules for antiderivatives

In order to construct a table of antiderivatives we may use a table of derivatives, only read in reverse order. As regards our course the following are the entries we are interested in (c is an arbitrary constant).

$$\begin{array}{c|l} f(x) & \displaystyle\int f(x)\,dx \\ \hline e^x & e^x + c \\ x^\alpha & \dfrac{x^{\alpha+1}}{\alpha+1} + c \quad \alpha \neq -1 \\ x^{-1} = \dfrac{1}{x} & \ln|x| + c \end{array} \tag{3}$$

If composite functions are involved, the same rules can be written as follows ($*$ stands for an arbitrary function).

$$\begin{array}{c|l} f(x) & \displaystyle\int f(x)\,dx \\ \hline e^*(*)' & e^* + c \\ *^\alpha(*)' & \dfrac{*^{\alpha+1}}{\alpha+1} + c \quad \alpha \neq -1 \\ *^{-1}(*)' = \dfrac{(*)'}{*} & \ln|*| + c \end{array} \tag{4}$$

The last rule in equation (4) is very important; its content can be expressed as follows: the antiderivative of a quotient where the numerator is the derivative of the denominator is simply the natural logarithm of the absolute value of the denominator. The next group of rules concerns the antiderivative of a sum of two functions and of the product of a function by a constant. The following formulas, that are an immediate consequence of the corresponding formulas for derivatives, hold:

$$\int \big(f(x)+g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx; \tag{5}$$

$$\int \big(k\,f(x)\big)\,dx = k\int f(x)\,dx. \tag{6}$$

The content of these formulas can be summarized as: the antiderivative of the sum is the sum of antiderivatives and the antiderivative of the product of a constant times a function is the product of the constant times the antiderivative of the function. One can also say that sums and antiderivatives can be interchanged, and the same holds for products between constants and functions. These are called *linearity properties* of the antiderivatives.

Some examples will clarify how to proceed. In the examples we will not write explicitly the additive constant.

— $\displaystyle\int \frac{3x^2}{x^3+1}\,dx = \int \frac{(x^3+1)'}{x^3+1}\,dx = \ln|x^3+1|$.

— $\displaystyle\int 2x\,e^{x^2}\,dx = \int e^{x^2}(x^2)'\,dx = e^{x^2}$.

— $\displaystyle\int x\,e^{x^2}\,dx = \int \frac{2}{2}x\,e^{x^2}\,dx = \frac{1}{2}\int 2x\,e^{x^2}\,dx = \frac{1}{2}e^{x^2}$: this example means that "if a *constant* lacks, you can always fix things in order to proceed". Observe also that in this case we have calculated the antiderivative of a product, but no rule for the antiderivative of a product has been used.

— A difficult situation! $\displaystyle\int \frac{1}{x\ln x}\,dx = \int \frac{1/x}{\ln x}\,dx = \int \frac{(\ln x)'}{\ln x}\,dx = \ln|\ln x|$.

— Also a difficult situation! $\displaystyle\int \frac{2x}{\sqrt{x^2+1}}\,dx = \int (x^2+1)^{-1/2}(x^2+1)'\,dx = \frac{(x^2+1)^{-1/2+1}}{-1/2+1} = \dots$

3 Integration by parts

We already know that the derivative of the product of two functions f and g is *not the product of derivatives*, but, at least, if one can calculate the derivative of each function a well formula allows us to calculate the derivative of the product of the two functions. Nothing similar holds for antiderivatives. An example is the following integral

$$\int \frac{e^x}{x}\,dx = \int \frac{1}{x}e^x\,dx\ :$$

although the antiderivatives of $1/x$ and e^x are well known (ande very simple!), there is no elementary way to compute the antiderivative of their product. Mathematicians say that an antiderivative of this function is a non elementary function, known by the name *Exponential integral function*.

We have also already met products of two functions and seen that the problem can be solved by means of the basic rules already considered, for instance the integral

$$\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + c.$$

However form the derivation rule of the product of two functions a rule for the antiderivative of the product can be obtained: it is called the formula of *integration by parts*. We'll write this formula with a little different notation than that of the textbook, but obviously with the same significance.

Let f and g be two functions and suppose we can calculate the antiderivative of *one of the two*, say f. If we denote one the antiderivatives of f by F, the following formula holds:

$$\int \big(f(x)\cdot g(x)\big)\,dx = \big(F(x)\cdot g(x)\big) - \int \big(F(x)\cdot g'(x)\big)\,dx. \tag{7}$$

Observe that in this formula the antiderivative of g is *never* used, while the antiderivative of f is used *two times*. Also note that this formula is useful only if one can calculate the last indefinite integral: we can say that this formula does not immediately solve the problem, rather it transforms an integral in another (that can also be not computable, or much more difficult than the previous one!). Some examples will clarify the problem.

Example 1. *Compute*

$$\int x\mathrm{e}^x \,\mathrm{d}x.$$

Using formula (7) we obtain

$$\int x\mathrm{e}^x \,\mathrm{d}x = \frac{x^2}{2} - \int \frac{x^2}{2}\mathrm{e}^x \,\mathrm{d}x,$$

and the last integral is much more complex than the first one. Let's try to reverse the order of the two functions.

$$\int x\mathrm{e}^x \,\mathrm{d}x = \int \mathrm{e}^x\, x \,\mathrm{d}x = \mathrm{e}^x\, x - \int \mathrm{e}^x \cdot 1 \,\mathrm{d}x = \mathrm{e}^x\, x - \mathrm{e}^x,$$

and the problem is solved!

Example 2. *Compute*

$$\int x^2\mathrm{e}^x \,\mathrm{d}x.$$

Using formula (7) we obtain, after reversing order,

$$\int \mathrm{e}^x\, x^2 \,\mathrm{d}x = \mathrm{e}^x\, x^2 - \int \mathrm{e}^x\, 2x \,\mathrm{d}x = \mathrm{e}^x\, x^2 - 2\int \mathrm{e}^x\, x \,\mathrm{d}x = \mathrm{e}^x\, x^2 - 2\big(\mathrm{e}^x\, x - \mathrm{e}^x\big),$$

and the problem is solved through two successive uses of the integration by parts.

Example 3. *Compute*

$$\int \ln x \,\mathrm{d}x.$$

We have

$$\int \ln x \,\mathrm{d}x = \int 1\cdot \ln x \,\mathrm{d}x = x\ln x - \int x\cdot\frac{1}{x} \,\mathrm{d}x = x\ln x - \int 1 \,\mathrm{d}x = x\ln x - x\,.$$

This last example is very important for our course, in that it allows us to compute the antiderivative of the natural logarithm function.

4 The definite integral

The kind of definite integral we will propose in these pages (which is the one presented in the textbook) is the simplest possible one and concerns only continuous functions. It is mainly based on the antiderivative. It is called *Newton-Leibniz integral*.

Several other kinds of integrals are considered by mathematicians, in order to allow integration of very discontinuous functions: the most important among them is the *Riemann integral*, while a very complete theory is *Lebesgue's* theory of integration. However for continuous functions all theories give the same results as the Newton-Leibniz integral. We only mention that which is the definition of definite integral in our presentation, it becomes the *Fundamental theorem of calculus* in Riemann's theory, and with this name it is discussed in the greatest part of analysis textbooks.

Given a function f, continuous on an interval D of the real line, an antiderivative F of f, and two numbers a and b belonging to D, we *define* the *definite integral* of f with *lower limit* a and *upper limit* b (or simply from a to b) to be the difference $F(b)-F(a)$. In formulas

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{8}$$

The difference $F(b)-F(a)$ is also written in one of the following ways:

$$F(b)-F(a)=\Big[F(x)\Big]_a^b=\Big|_a^b F(x)=F(x)\Big|_a^b .$$

Note that in this definition b must not be greater that a. Note also that, despite the similarity of symbols, a definite integral is something completely different from an indefinite integral: the indefinite integral is a set of functions, the definite integral is a number.

If the antiderivative F of f is known, the calculation of a definite integral is only a matter of substitution.

Let us consider a simple example.

Example 4. *Given the function $f(x)=2x+1$, compute the definite integral of f between 1 and 2 and try to give a geometrical interpretation of the result.*

An antiderivative of this function is very simple: $F(x)=x^2+x$. So we obtain

$$\int_1^2 (2x+1)\,\mathrm{d}x=\Big[x^2+x\Big]_1^2=(4+2)-(1+1)=4.$$

By considering figure 1 one can easily conclude that the value of this definite integral is exactly the area of the shaded rectangle trapezoid.

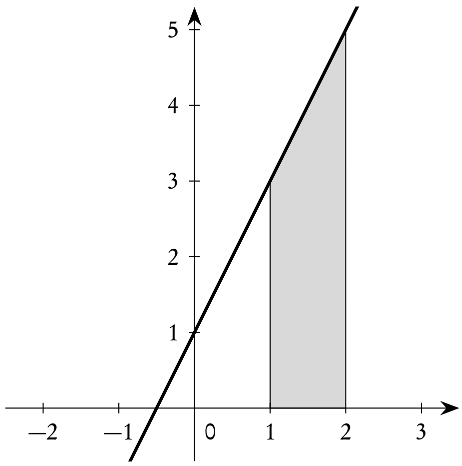

Figure 1 *The integral of example 4*

In general the geometric interpretation(1) of a definite integral is as follows.

Consider a (continuous) function f and two numbers a and b belonging to the domain. Plot a rough graph of the function. Then consider the oriented closed loop from $(a,0)$ to $(b,0)$ on the x axis, then vertically to $(b,f(b))$, then to $(a,f(a))$ following the graph, then finally vertically again to $(a,0)$. This closed loop identifies a group of bounded regions in the cartesian plane, some of which with the edge path in a counterclockwise direction, some other with the edge path in a clockwise direction. Then one can prove that

$$\int_a^b f(x)\,\mathrm{d}x$$

always represents the sum of the areas of the regions with the edge path in a counterclockwise direction, minus the the sum of the areas of the regions with the edge path in a clockwise direction. See figures 2 and 3.

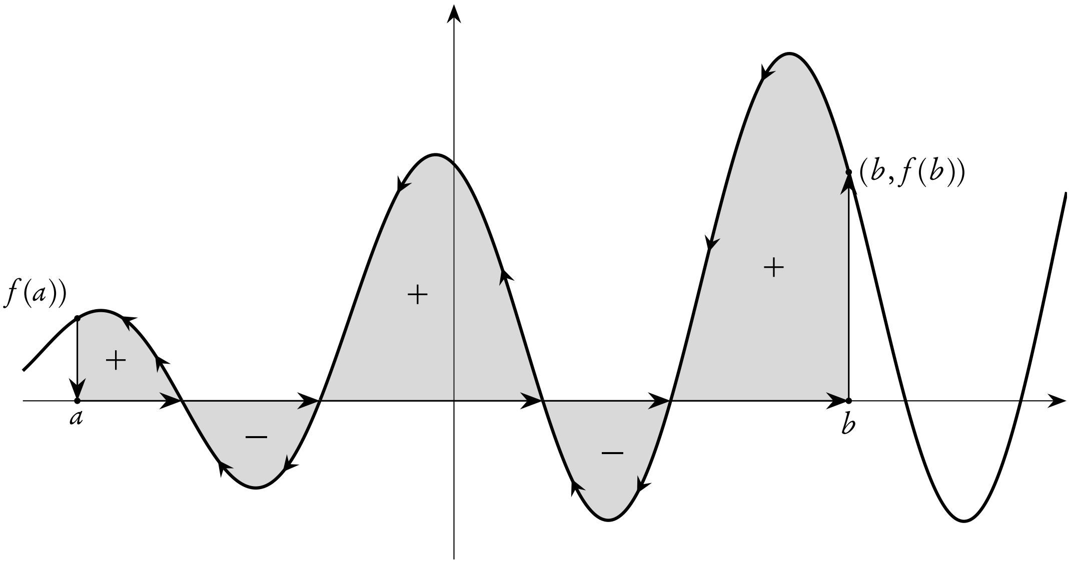

Figure 2 *Geometrical interpretation of the definite integral* $\int_a^b f(x)\,\mathrm{d}x$*: positive and negative contributions to the integral*

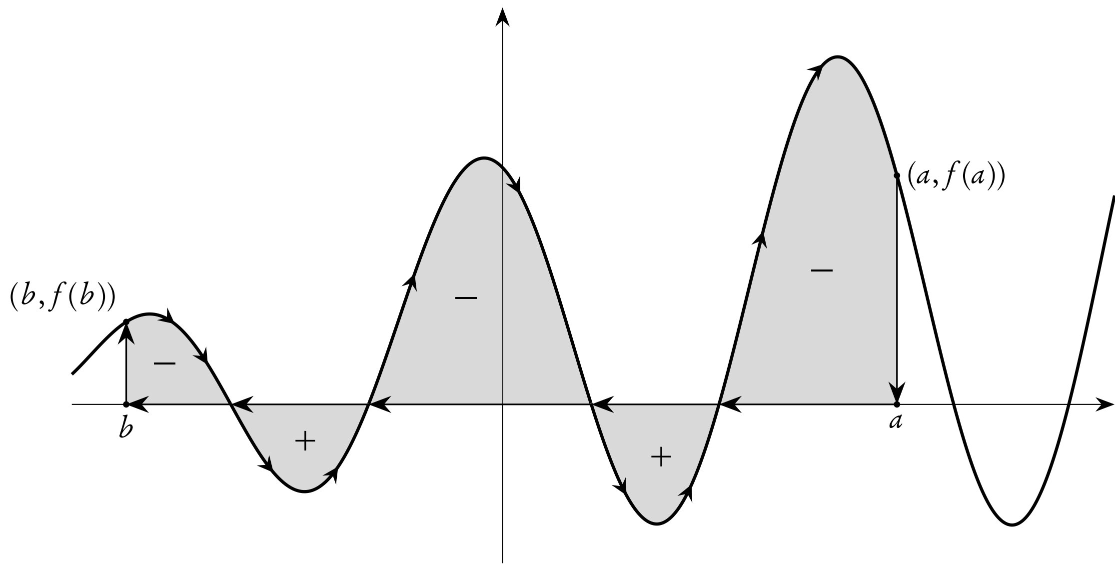

Figure 3 *The same integral as in figure 2, but with* a *and* b *interchanged: positive and negative contributions to the integral are exactly the opposite of the previous case*

In particular one can conclude that if only one region is identified, the integral represents the area of that region or the opposite of that area depending on the direction of the edge path of that region.

A frequently required application of definite integrals is the calculation of the area between the graphs of two (continuous) functions and two vertical lines $x = a$ and $x = b$, with, in this case, $a < b$. We suppose that one of the two functions is always above the other in the interval $[a, b]$: let's call f_{up} and f_{down} the two functions. It is easy enough to prove, using the geometrical interpretation given above,

[1]The geometric interpretation given here is, in our opinion, the simplest one. However it is by no means common in textbooks.

that
$$\text{Area} = \int_a^b \Big(f_{\text{up}} - f_{\text{down}} \Big) \mathrm{d}x$$
whatever the position of the functions in the cartesian plane. See figure 4.

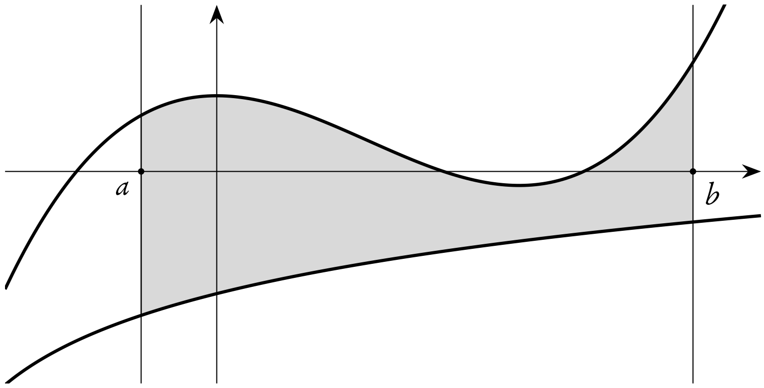

Figure 4 *The region between the graphs of two functions*

Among the properties of the definite integral we recall the following, urging students to demonstrate them, on the base of the definition of definite integral.

- $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0.$
- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x.$
- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$, whatever the order of the given points a, b, c.

5 Improper integrals

The formula

$$\int_a^b f(x)\,\mathrm{d}x \tag{9}$$

makes sense only if a and b are real numbers that belong to the domain of the continuous function f. It is possible, and useful for applications, to extend this definition in two ways:

1. by allowing one of the limits, or even both, of the integral is infinite: we'll speak of *infinite intervals of integration*;
2. by considering functions defined on half-open intervals and with the property that the limit as x tends to one of the bounds of the interval the function tends to $\pm\infty$: we'll speak of *integrals of unbounded functions*.

We give here the definitions for the most important cases.

1. If f is a function defined for all $x > a$, then

$$\int_a^{+\infty} f(x)\,\mathrm{d}x = \lim_{b\to+\infty} \int_a^b f(x)\,\mathrm{d}x. \tag{10}$$

2. If f is a function defined for all $x < b$, then

$$\int_{-\infty}^{b} f(x)\,\mathrm{d}x = \lim_{a\to-\infty} \int_{a}^{b} f(x)\,\mathrm{d}x. \tag{11}$$

3. If f is a function defined in a half-open interval $]a, b]$, then

$$\int_{a}^{b} f(x)\,\mathrm{d}x = \lim_{h\to a^+} \int_{h}^{b} f(x)\,\mathrm{d}x. \tag{12}$$

4. If f is a function defined in a half-open interval $[a, b[$, then

$$\int_{a}^{b} f(x)\,\mathrm{d}x = \lim_{h\to b^-} \int_{a}^{h} f(x)\,\mathrm{d}x. \tag{13}$$

The limits considered may not exist and, if they exist, they can be $\pm\infty$. If the limits exist and are finite we say that the integrals *converge* or that the function is *integrable* in the given interval. Nothing changes as regards the geometric interpretation of the integrals as areas with a plus other minus sign.

Refer to the textbook for examples.