Università Ca' Foscari di Venezia - Dipartimento di Economia - A.A.2016-2017

Mathematics (Curriculum Economics, Markets and Finance)

A brief introduction to limits

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1 Examples and definitions based on them

Example 1. *Given the function*

$$f(x)=\frac{e^x-1}{x}, \tag{1}$$

consider the problem of studying its behaviour when x (the independent variable) is close to 0.

The (natural) domain of this function is $\mathbb{R}\setminus\{0\}$, so it is not possible to find its value when x is exactly 0. Nevertheless there is no problem if we choose x closer and closer to 0. In figure 1 the graph(1) of this function near 0 is plotted.

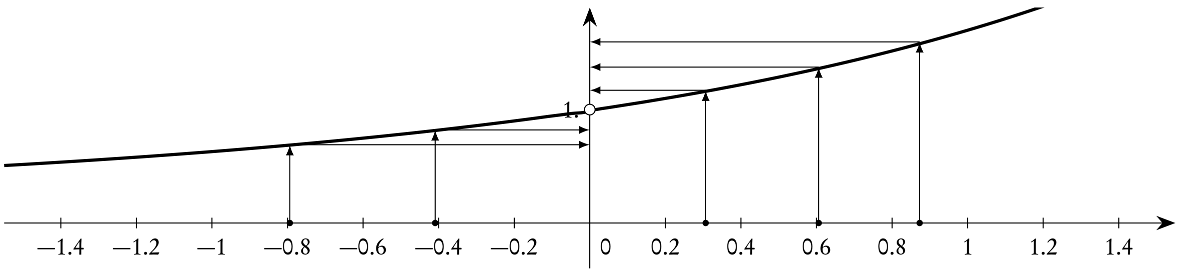


Figure 1 *The function* $f(x)=(e^x-1)/x$*, near* 0

As you can see, as x gets closer and closer to 0, the corresponding value of $y=f(x)$ gets closer and closer to 1. You can check this fact using a pocket calculator. A small circle is used to indicate that the point $(0,1)$ does not belong to the graph: the value $f(0)$ does not exist! We shall summarize this fact by saying that "*the limit, as x gets closer and closer to* 0*, of* $f(x)$*, is* 1", and write:

$$\lim_{x\to 0}\frac{e^x-1}{x}=1. \tag{2}$$

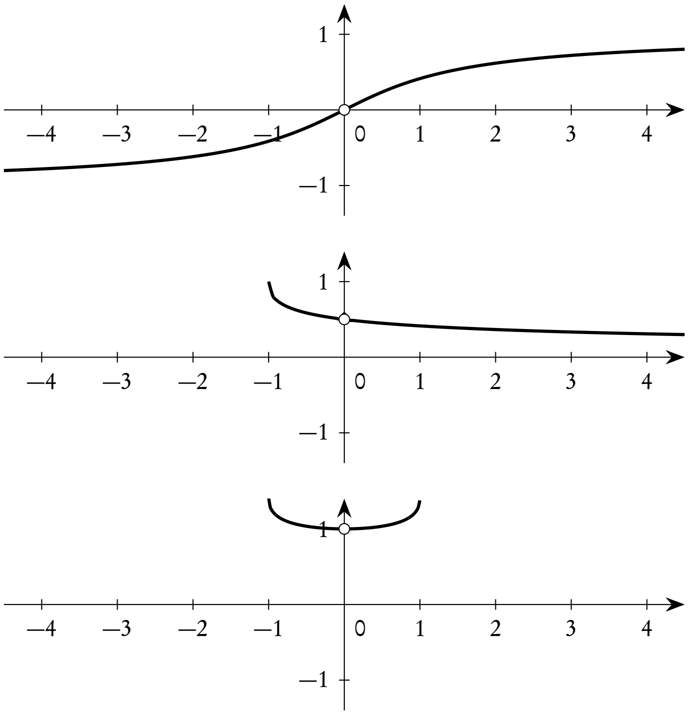
Observe that, as x approaches 0, also the numerator of the fraction that defines f approaches 0, so the fraction itself approaches the "form" 0/0. Many people say that the fraction 0/0 is *indeterminate*: this is completely wrong. A fraction whose denominator is 0 is *not defined* at all, whatever the numerator.

(1) In all the graphs pay close attention to the units used on the horizontal and vertical axis: they are usually different!

It is correct to say, as we'll see, that, as x approaches 0, the function approaches a form that makes the calculation of the limit very difficult. The problem is that we can have different functions that approach the same form 0/0 as x approaches a particular value: despite this fact different functions can approach completely different values. It is in this sense that the word *indeterminate form* must be used. For example it's easy to verify using a pocket calculator or a simple program of computer graphics the following facts, where the functions have different limits, although all are in the form 0/0.

$$\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{x}=0;\qquad \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{1}{2};\qquad \lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=1.$$

For the sake of completeness the graphs are reproduced below.

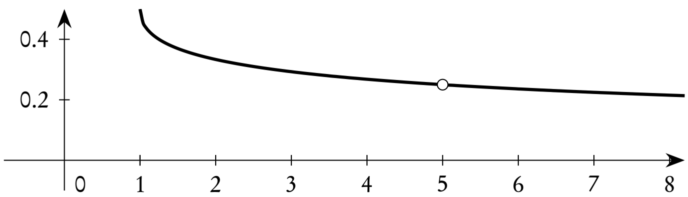


Do not think that the value 0 for the x is important: what matters in this example is the fact that the value under consideration does not belong to the domain. For the function, whose graph is reproduced below,

$$g(x)=\frac{\sqrt{x-1}-2}{x-5}$$

the problem arises when one tries to get closer and closer to 5 with the value of x. As usual, using a pocket calculator or a program of computer graphics you can find that

$$\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x-5}=\frac{1}{4}.$$



Example 2. *Given the function*

$$f(x)=\begin{cases} x^2, & \text{if } x<1; \\ 2, & \text{if } x=1; \\ x-1, & \text{if } x>1 \end{cases}, \tag{3}$$

consider the problem of studying its behaviour when x is close to 1.

As the natural domain of this function is $\mathbb{R}$, it is okay to calculate $f(1)$, and you get $f(1)=2$, but the graph in figure 2 (and the definition itself!) suggests that the behaviour of the function near 1 is not common at all!

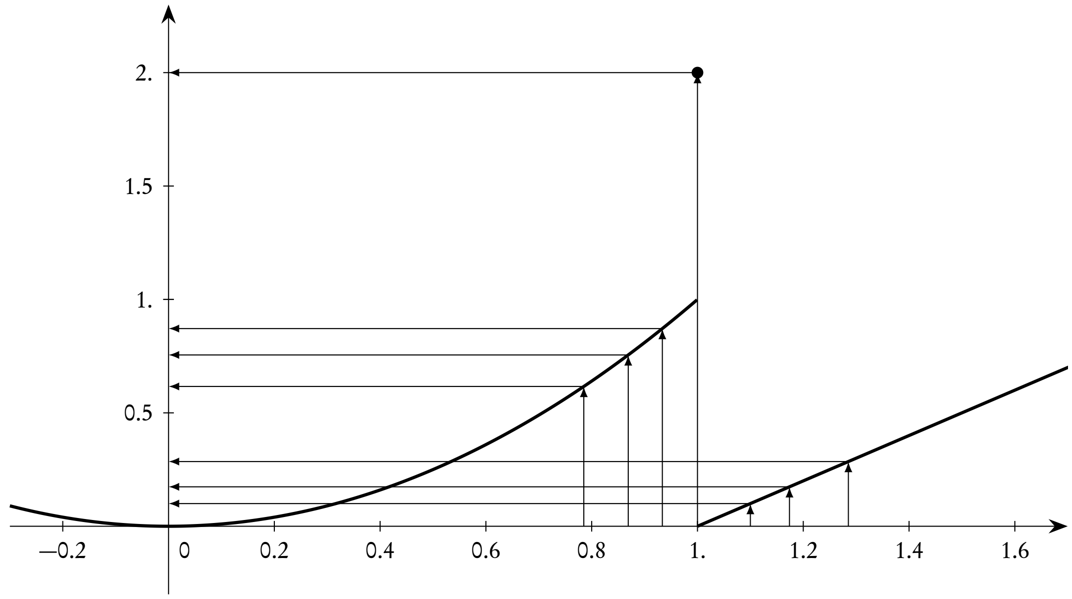


Figure 2 *The function of example 2, near* 1

The graph in figure 2 suggests that as x gets closer and closer to 1 from the left, the function gets closer and closer to 1, while if you procede the same way, but from the right, the function approaches 0; furthermore if you reach exactly 1 both from the right and left the function suddenly jumps to 2, and this is absolutely correct because $f(1)=2$, as already mentioned. We shall summarize these facts by saying that "*the limit, as x gets closer and closer to* 1 *from the left, of $f(x)$, is* 1, *while the limit as x gets closer and closer to* 1 *from the right, of $f(x)$, is* 0" and write:

$$\lim_{x\to 1^-} f(x)=1, \qquad \lim_{x\to 1^+} f(x)=0. \tag{4}$$

We call these limits *one-sided* limits, the first *from below* or *from the left*, the second *from above*, or *from the right*. They are also called *left limit* and *right limit*, respectively.

In this example, unlike example 1, the point 1 *belongs* to the domain of the function, but observe that the value of $f(1)$ has no importance at all for the left limit or right limit. If we had defined $f(1)=3$, or any other value, nothing would change as regards these limits. Even if we had not defined the value of $f(x)$ corresponding to $x=1$, nothing would change: the value of a function as x matches exactly a particular point is not important at all as far as the concept of limit is concerned: sometimes you can calculate the value of a function corresponding to a particular point, sometimes you can't, but there is no difference in the two cases when considering the behaviour of a function as x gets closer and closer to that point.

In the following we'll be interested in the problem: knowing the left and right limits, what can be said about the limit? The answer is a specific theorem[2]: *a limit exists if and only if left and right limits both exist and are equal.*

Example 3. *Given the function*

$$f(x) = x^2 - x, \tag{5}$$

consider the problem of studying its behaviour when x is close to 2.

The natural domain of this function is $\mathbb{R}$, so it is okay to calculate $f(2)$, and you obtain $f(2) = 2$. In this case it is immediate to conclude, either by using a pocket calculator or a graph produced by a computer software that

$$\lim_{x \to 2} f(x) = 2.$$

The graph is reproduced in figure 3.

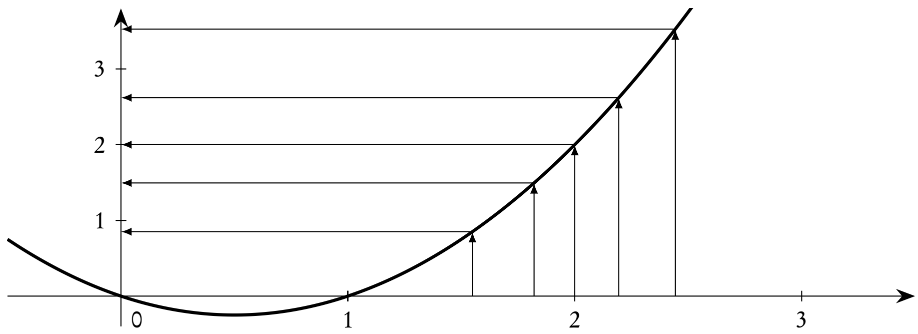


Figure 3 *The function of example 3*

In this case both the left and right limit exist, and they are equal, so the limit itself exists. Furthermore the point 2 belongs to the domain of the function and the limit is exactly equal to the value $f(2)$ (unlike what happened in example 2). This situation is particularly important in applications: we shall call a function with this property a *continuous* function at the point under consideration.

Since this concept is of utmost importance we give a more formal definition.

Definition 1. *A function f, with domain D, is* continuous *at a point a if*

- *a belongs to the domain;*
- *the limit of f as $x \to a$ exists;*
- *the limit is exactly the value of the function at a.*

The definition may be shortly written in formulas:

$$\lim_{x \to a} f(x) = f(a). \tag{6}$$

If one knows in advance that a function is continuous at a particular point a, the calculation of the limit as $x \to a$ is immediate: you must simply compute $f(a)$, that is the calculation of the limit is only a matter of substitution.

The most important result in this regard is the following.

Theorem 2. *All elementary functions are* continuous *at all points of their domain.*

[2] We'll usually not consider in the future the proofs of theorems. You can find a very limited number of them in your textbook; if interested you can consult any analysis textbook.

Naturally this theorem is useful only if we know what the *elementary* functions are. As regards our course they are all the functions you have met in the first five chapters of your textbook, except for the piecewise defined functions. This means that we'll have some problem in calculating limits in the following cases:

— functions piecewise defined on the base of elementary functions, as x tends to one of the points of separation between an expression of the function and another;
— elementary functions as x tends to a point not in the domain;
— elementary functions as x "grows without limits" (see the following example 4).

We'll study some simples techniques to deal with these situations.

Example 4. *Given the function*

(7) $$f(x)=\ln x,$$

consider the problem of finding its behaviour as x "grows without bounds".

As now usual it's simple to conclude that as x grows without bounds, also $f(x)$ grows without bounds. The plot in figure 4 illustrates this situation.

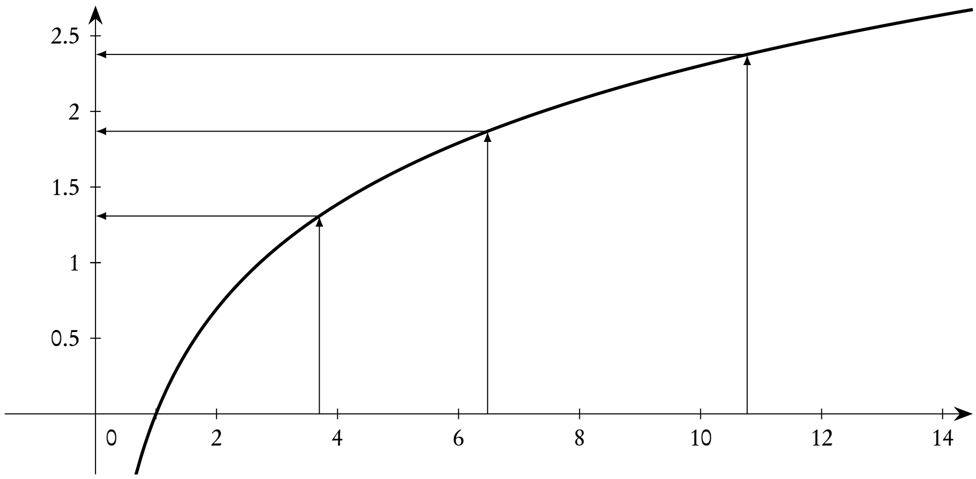


Figure 4 *The function of example 4*

We shall summarize this fact by saying that "*the limit, as x tends to $+\infty$, of $f(x)$, is $+\infty$*", and write

(8) $$\lim_{x\to+\infty} \ln x = +\infty.$$

In the same way, and we invite students to find examples taken from the theory of elementary functions, one can consider the following situations.

(9) $$\lim_{x\to+\infty} f(x)=-\infty, \quad \lim_{x\to-\infty} f(x)=+\infty, \quad \lim_{x\to-\infty} f(x)=-\infty.$$

Example 5. *Given the function*

(10) $$f(x)=\frac{x+1}{2x},$$

consider the problem of finding its behaviour as x "grows without bounds".

Again, with the repeatedly used techniques, it's simple to conclude that as x grows without bounds, $f(x)$ approaches 1/2.

We will summarize this fact by saying that "*the limit, as x tends to* $+\infty$, *of* $f(x)$, *is* 1/2", and write

$$\lim_{x\to+\infty} f(x) = \frac{1}{2}. \tag{11}$$

The graph of figure 5 illustrates this situation.

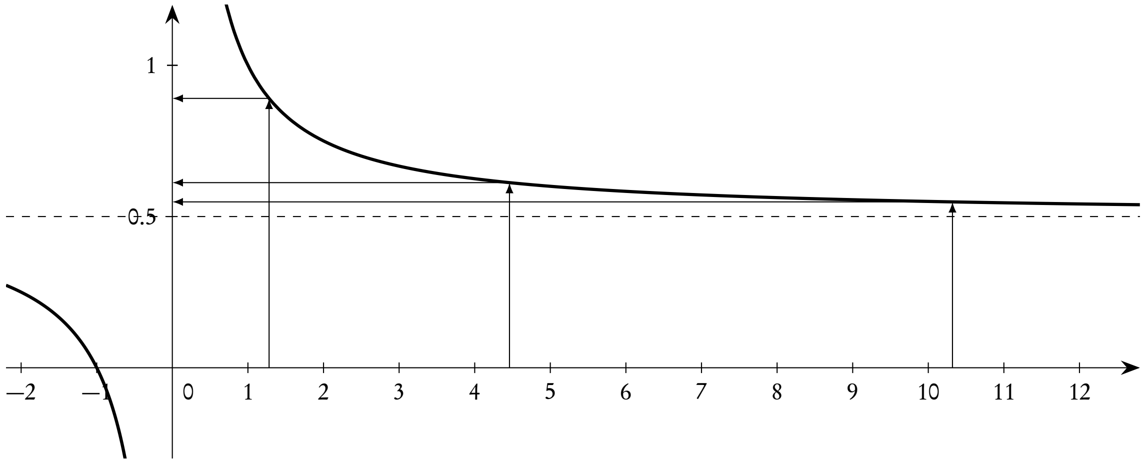


Figure 5 *The function of example 5*

If you remember the properties of the exponential function with base "e" (natural exponential function), you can easily conclude that

$$\lim_{x\to-\infty} e^x = 0. \tag{12}$$

The plot in figure 6 illustrates this situation.

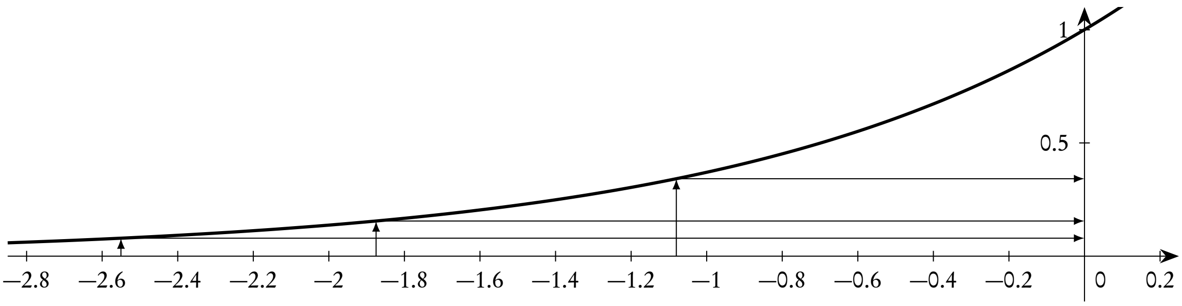


Figure 6 *The natural exponential function and its limit as* $x \to -\infty$

By closely examining figures 5 and 6 one can see that the graph of the functions approaches the lines $y = 1$ and $y = 0$ (the x-axis) as x tends to $+\infty$ or $-\infty$, respectively. Lines of this kind are called *horizontal asymptotes*: these two lines represent what is called the asymptotic behaviour of the function. Finding the asymptotic behaviour of a function is very important in many applications, also in economic ones.

Example 6. *Given the function*

$$f(x) = \frac{1}{x^2}, \tag{13}$$

consider the problem of finding its behaviour as x gets closer and closer to 0.

In this case, using the now usual techniques, you can conclude that, as x gets closer and closer to 0, $f(x)$ grows without bounds. We can summarize this fact by writing

$$\lim_{x\to 0}\frac{1}{x^2}=+\infty. \tag{14}$$

The plot in figure 7 illustrates this situation.

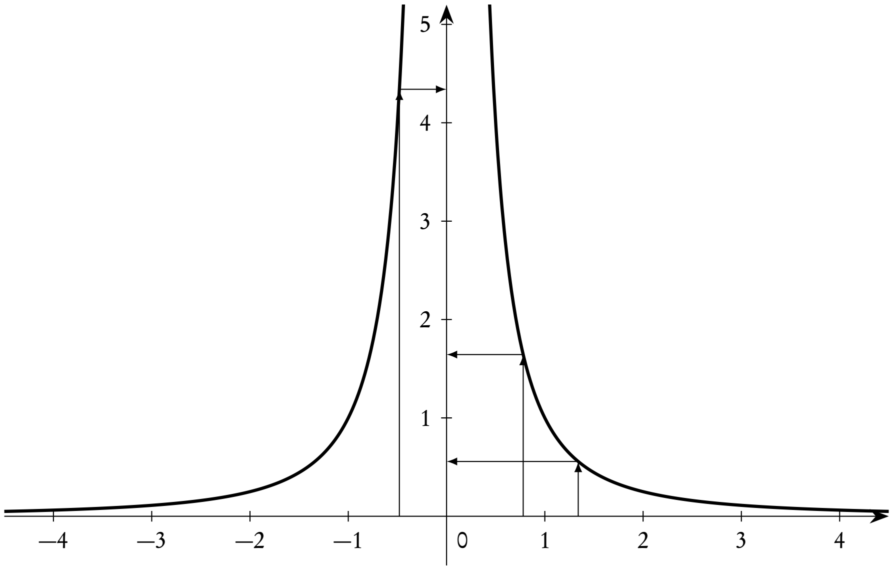


Figure 7 *The function of example 6*

Using the graphs of elementary functions that you already know, and the previously defined concepts, try to understand the meaning of the following formulas:

$$\lim_{x\to 0^-}\frac{1}{x}=-\infty,\qquad \lim_{x\to 0^+}\frac{1}{x}=+\infty,\qquad \lim_{x\to 0^+}\ln x=-\infty. \tag{15}$$

Lines like $x=0$ (the y-axis) in figure 7 are called *vertical asymptotes*: the graph approaches this line as x gets closer and closer to 0, both from below and above. In the case of the natural logarithmic function the vertical line $y=0$ is also a vertical asymptote, although the graph approaches this line only from above.

Example 7. *Use computer graphics to plot the graph of the function*[(3)]

$$f(x)=\sin\left(\frac{1}{x}\right). \tag{16}$$

Can you conclude something about the problem of finding its behaviour as x gets closer and closer to 0*?*

This function is often called "the analysis monster": there is no way to plot a significant graph near 0, regardless of the zoom level.

While figure 8 suggests a simple behaviour, the subsequent figures, where a different zoom level near 0 is used, prove that this is just an impression at first sight.

[3] This example uses a trigonometric function. In our course we'll not be interested in such (that would anyway be interesting!) functions, so do not worry: only use a computer to plot the graph.

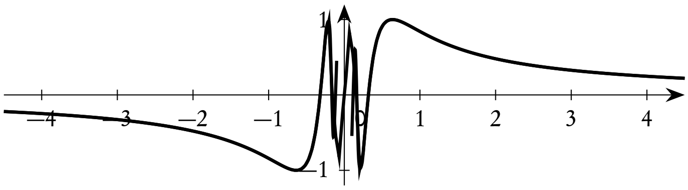


Figure 8 *The function of example 7*

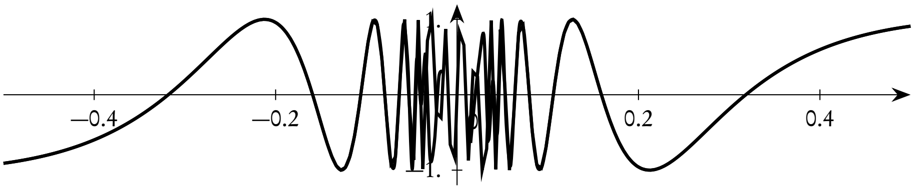


Figure 9 *The function of example 7, with a different zoom level*

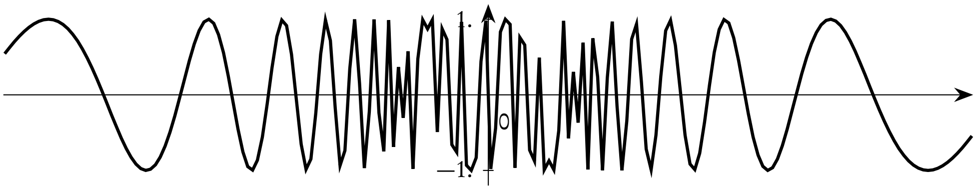


Figure 10 *The function of example 7, with a still different zoom level*

In this case computer graphics is of no help: with other considerations one can conclude that

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \quad \nexists: \tag{17}$$

there is not a defined behaviour of this function as x gets closer and closer to 0.

Although this function is a "monster function", in nature (and in many economic situations) the behaviour is even stranger!

2 Summary of the definitions introduced

If f is a function with domain D (and codomain $\mathbb{R}$) and a is a point for which it makes sense to consider the problem of finding the behaviour of $f(x)$ as x gets closer and closer to a, we have defined the concept of *limit, as x tend to a, of $f(x)$*.

The same can be done if we consider the problem of finding the behaviour of the function as x grows without bounds (tends to $+\infty$) or decreases without bounds (tends to $-\infty$).

The examples considered in section 1 summarize the situations of interest in our course. We report three cases that interest us more.

1. If a is a point in the domain of a function f, when

$$\lim_{x\to a} f(x) = f(a)$$

the function is said to be *continuous* at a. All elementary functions are continuous at all points of their natural domain. At the level of our course the easiest way to construct non continuous

functions is to use piecewise defined functions: these functions are of utmost importance in economic situations.
Roughly speaking a function is continuous if its graph can be plotted without leaving one's pencil off the paper, that is if the graph has no jumps. However, this property is only a first sight property: there are continuous functions whose graph cannot be plotted at all!

2. If $a \in \mathbb{R}$ is a point belonging or not to the domain of a function f, when

$$\lim_{x \to a} f(x) = \pm\infty$$

the vertical line $x = a$ is called a *vertical asymptote* of the function.

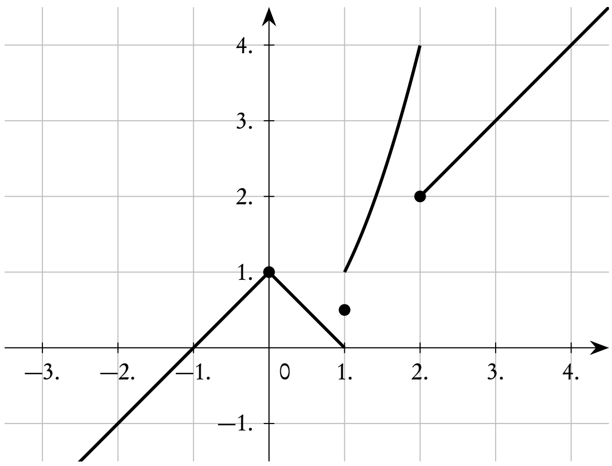
3. If a function f is defined for x close to $+\infty$, or to $-\infty$, when

$$\lim_{x \to +\infty} f(x) = l, \quad \text{with } l \in \mathbb{R}$$

the horizontal line $y = l$ is called a *horizontal asymptote* of the function. The same definition holds also when $+\infty$ is replaced by $-\infty$.

3 First exercises

Exercise 1. *Let f be the function whose graph is plotted in the figure below.*



Answer the following questions.

a) What are the values of the function at $x = -1$, $x = 0$, $x = 1$, $x = 2$?

b) Calculate the following limits.

$$\lim_{x \to -1} f(x), \quad \lim_{x \to 0} f(x), \quad \lim_{x \to 1} f(x), \quad \lim_{x \to 2} f(x).$$

c) Are there points where the function is not continuous?

Solution. a) The requested values are, in order, 0, 1, 1/2, 2.

b) We may easily conclude that $\lim_{x \to -1} f(x) = 0$;
$\lim_{x \to 0} f(x) = 1$ (both from below and from above);
$\lim_{x \to 1} f(x) \quad \nexists$, because the left limit is 0 and the right limit is 1 (both these limits are different from $f(1)$);
$\lim_{x \to 2} f(x) \quad \nexists$, because the left limit is 4 and the right limit is 2 (in this case the right limit equals the value of the function at 2).

c) As a consequence of previous results the function is not continuous at $x=1$ and $x=2$. □

Exercise 2. *Consider the function*

$$f(x)=\begin{cases} x^2-ax+1, & \text{if } x\le 1;\\ -x+2, & \text{if } x>1; \end{cases},$$

where a is a real parameter. Find the value of a for which the function is continuous. Plot the corresponding graph.

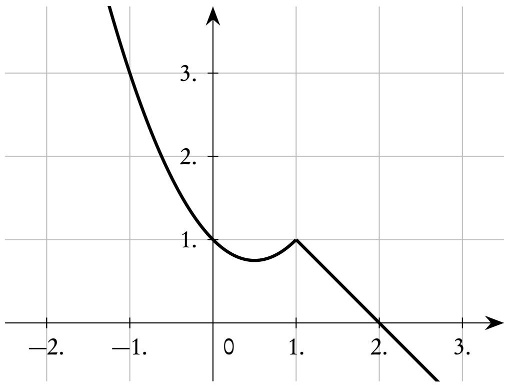
Solution. This function is everywhere continuous except perhaps at the point $x=1$ where we switch between the two expressions of the function. As the two functions that are used to make up f are elementary functions, we can conclude that

$$\lim_{x\to 1^-} f(x)=1-a+1=2-a, \quad \lim_{x\to 1^+} f(x)=1.$$

In order that the function is continuous the two limits from below and from above must be equal:

$$2-a=1 \quad \Rightarrow \quad a=1.$$

Plotting the graph is now a routine operation. □



Exercise 3. *Given the function*

$$f(x)=\begin{cases} x^2, & \text{if } x\le 1;\\ ax+1/2, & \text{if } 1<x<2;\\ -x+b, & \text{if } x\ge 2; \end{cases},$$

where a and b are real parameters, find the values of a and b for which the function is everywhere continuous. For these values of the parameters plot the graph of the function.

Solution. This function is everywhere continuous except perhaps the points $x=1$ and $x=2$ where we switch between the different expressions of the function. As the three functions that are used to make up f are elementary functions, we can conclude that

$$\lim_{x\to 1^-} f(x)=1, \quad \lim_{x\to 1^+} f(x)=a+1/2,$$

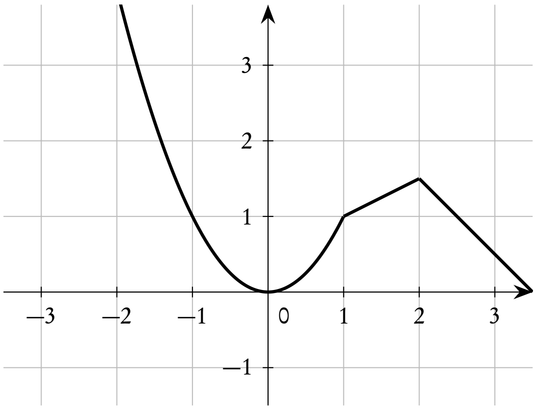
and

$$\lim_{x\to 2^-} f(x)=2a+1/2, \quad \lim_{x\to 2^+} f(x)=-2+b.$$

In order that the function is continuous we must have

$$\begin{cases} 1=a+1/2\\ 2a+1/2=-2+b \end{cases} \quad \Rightarrow \quad \begin{cases} a=1/2\\ b=7/2 \end{cases}.$$

Plotting the graph is now a routine operation. □



4 Rules for limits

As regards our course, we are only interested in the calculation of simple limits. We summarize the rules that can be used. However, it is important to note that the calculation of limits is not easy at all, even in simple cases. Let's reconsider example 1 on page 1: there we have concluded, on the base of a graph obtained using computer graphics, that

$$\lim_{x\to 0}\frac{e^x-1}{x}=1.$$

The formal proof of this result is not easy at all, and involves a precise definition of the number "e" (Napier's number), definition that is beyond the scope of this course.

All the "rules" we present come as a result of a specific theorem: here we only summarize the thesis of these theorems.

First of all let's recall some basic limits, that are a simple consequence of the properties of the elementary functions involved.

- $\lim_{x\to\pm\infty} x=\pm\infty$.
- $\lim_{x\to+\infty} e^x=+\infty$.
- $\lim_{x\to-\infty} e^x=0$.
- $\lim_{x\to 0^+} \ln x=-\infty$.
- $\lim_{x\to+\infty} \ln x=+\infty$.

The last four cases will be briefly written as follows:

$$e^{+\infty}=+\infty,\quad e^{-\infty}=0,\quad \ln(0^+)=-\infty,\quad \ln(+\infty)=+\infty.$$

Occasionally we'll consider also exponentials or logarithms with base $a\neq e$. In these cases the following brief "formulas" hold.

1. If $a>1$ then

$$a^{+\infty}=+\infty,\quad a^{-\infty}=0,\quad \log_a(0^+)=-\infty,\quad \log_a(+\infty)=+\infty,$$

exactly as in the case where $a=e$.

2. If $0<a<1$ then

$$a^{+\infty}=0,\quad a^{-\infty}=+\infty,\quad \log_a(0^+)=+\infty,\quad \log_a(+\infty)=-\infty.$$

In order to simplify calculations with exponentials or logarithms it is often useful to change the base. The following formulas hold (they are a consequence of the definition of logarithms):

$$a^x = e^{x \ln a}, \quad \log_a x = \frac{\ln x}{\ln a}.$$

The second one is particularly important if one needs to calculate a logarithm with base different from "e" or from "10" using a pocket calculator: in fact only natural logarithms and logarithms with base 10 are usually implemented in pocket calculators.

Now suppose we have two functions f and g and we already know that

$$\lim_{x \to a} f(x) = l, \quad \lim_{x \to a} g(x) = m,$$

where l and m are real numbers, or

$$\lim_{x \to a} f(x) = \pm\infty, \quad \lim_{x \to a} g(x) = \pm\infty.$$

Sometimes in the following it will be important to consider separately the case where $l \neq 0$ or $m \neq 0$: we'll indicate explicitly this fact by writing $(l \neq 0)$, or $(m \neq 0)$.

We are interested in calculating the limit, as $x \to a$, of $f \pm g$, $f \cdot g$, f/g. For example for the sum of the two functions the following theorem holds:

Theorem 3. *If*

$$\lim_{x \to a} f(x) = l, \quad \lim_{x \to a} g(x) = m,$$

where l and m are real numbers, then[4]

$$\lim_{x \to a} \big(f(x) + g(x)\big) = l + m.$$

Instead of writing the theorem in this complete form, we'll simply write: "$l + m$ is calculated as usual". With this in mind we know write the "rules" that concern us.

— $l \pm m$ is calculate as usual.

— $l \cdot m$ is calculated as usual.

— $\dfrac{l}{(m \neq 0)}$ is calculated as usual.

— $\dfrac{(l \neq 0)}{0} = \infty$, with the proper sign, as determined by the rule of signs[5].

— $l \pm \infty = \pm\infty$.

— $+\infty + \infty = +\infty$.

— $-\infty - \infty = -\infty$.

— $(l \neq 0) \cdot \infty = \infty$, with the proper sign.

[4]Observe that this theorem can be stated as follows: *The limit of the sum of two functions is the sum of the limits*. This means that the two operations of *limit* and *sum* can be interchanged. It is always important to know if it is possible or not to change the order when two operations are in series.

[5]Pay close attention to this fact. We have *not* said that a quotient of a number different from 0 with 0 is defined: division by 0 is always prohibited, there is nothing new as far as this problem is concerned. The given equality means exactly the following: *If you have two functions f and g, such that the limit of f, as x tends to a, is different from* 0*, while the limit of g, as x tends to a, is* 0*, then the limit of the quotient is ∞, with the proper sign*.

- $\infty \cdot \infty = \infty$, with the proper sign.
- $\dfrac{l}{\pm\infty} = 0$.
- $\dfrac{\infty}{m} = \infty$, with the proper sign.
- $(\infty)^{\text{positive power}} = \infty$, with the proper sign.
- $(\infty)^{\text{negative power}} = 0$.

The following cases can't be treated by a theorem:

1. $\infty - \infty$.
2. $0 \cdot \infty$.
3. $\dfrac{0}{0}$.
4. $\dfrac{\infty}{\infty}$.

This situations are called *indefinite forms*.

The reason for this particular behaviour is that in such cases the result may be different for different functions. If you reconsider example 1 on page 1, we have already met this problem: in five cases we have considered the limit of the quotient of two functions both having 0 as limit, and the result changes from one situation to another. It is in this sense that one can say that $0/0$ is *indefinite*. Pay close attention (see also the footnote 5) to this fact. It is *incorrect* to say that the division of 0 by 0 is *indefinite* or *undetermined*: the division of 0 by 0 is simply *undefined*, because the denominator of a fraction can't be 0. What happens in limits is an entirely different thing, that can be summarized as follows: *If you have two functions f and g, and both have the same limit 0 as x tends to a, then it is impossible to conclude,* without further investigation, *what the limit of the quotient is*. Observe that we have said: "*without further investigation*". In fact, using appropriate techniques, it is always possible to find what the limit is, but the problem may be very very difficult, as have already pointed out. So "indefinite form" does not mean that we can't calculate the limit, it only means that the calculation requires just "further investigation". In our opinion it would be better to call these situations "difficult forms", but, alas, we are forced to keep the official name.

5 Some simple technique

The following exercises show some useful technique for the calculation of limits, also in the case of indefinite forms. Only very simple situations will be considered.

Exercise 4. *Calculate*

$$\lim_{x\to 1^+} \frac{1}{\ln x}.$$

Solution. We have

$$\ln 1^+ = 0^+ \Rightarrow \lim_{x\to 1^+} \frac{1}{\ln x} = \frac{1}{0^+} = +\infty.$$

The symbol 0^+ is used to point out the fact that, as $x \to 1^+$, the function $\ln x$ tends to 0, but it is always positive. □

Exercise 5. *Calculate*

$$\lim_{x\to 1} \frac{x^3 - 1}{x - 1}.$$

Solution. We have
$$\lim_{x\to 1}\frac{x^3-1}{x-1}=\frac{0}{0},$$
so we get an indefinite form. The technique applied in this example is very common and may be summarized as follows.

While calculating a limit as $x \to a$, try to substitute the value of a to the x: if you do not obtain an indefinite form, you can apply the rules discussed in section 4, otherwise you must try some different technique (and usually the problem is very difficult!).

In the case under consideration you can observe that
$$x^3-1=(x-1)(x^2+x+1),$$
then
$$\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}}=\lim_{x\to 1}x^2+x+1=3.$$
This "simplification" is often useful in the case $0/0$. □

Exercise 6. *Calculate*
$$\lim_{x\to 2^+}\frac{1-x}{(x-2)^2}.$$

Solution.
$$\lim_{x\to 2^+}\frac{1-x}{(x-2)^2}=\frac{-1}{0^+}=-\infty.$$
□

Exercise 7. *Calculate*
$$\lim_{x\to 2^-}\frac{1-x}{(x-2)^2}.$$

Solution.
$$\lim_{x\to 2^-}\frac{1-x}{(x-2)^2}=\frac{-1}{0^+}=-\infty.$$
□

Exercise 8. *Calculate*
$$\lim_{x\to+\infty}x^3+2x^2+x+1.$$

Solution. By "substituting" $+\infty$ for x, and using the rules for limits, we obtain
$$\lim_{x\to+\infty}x^3+2x^2+x+1=(+\infty)^3+3(+\infty)^2+(+\infty)+1=+\infty.$$
□

Exercise 9. *Calculate*
$$\lim_{x\to-\infty}x^3+2x^2+x+1.$$

Solution. By "substituting" $+\infty$ for x we obtain
$$\lim_{x\to-\infty}x^3+2x^2+x+1=(-\infty)^3+3(-\infty)^2+(-\infty)+1=-\infty+\infty-\infty+1,$$
so we can't conclude (indefinite form). If we factor out the greatest power of x (that is x^3) we obtain:
$$\lim_{x\to-\infty}x^3+2x^2+x+1=\lim_{x\to-\infty}x^3\left(1+\frac{2}{x}+\frac{1}{x^2}+\frac{1}{x^3}\right).$$
Now by substituting again and remembering the rules for limits we obtain:
$$(-\infty)^3\left(1+\frac{2}{-\infty}+\frac{1}{(-\infty)^2}+\frac{1}{(-\infty)^3}\right)=-\infty(1+0+0+0)=-\infty.$$
This technique is useful for polynomials or rational fractions as $x\to\pm\infty$. □

Exercise 10. *Calculate*

$$\lim_{x\to+\infty} \frac{x^2-x-1}{3x^3-2x-3}.$$

Solution. By factoring out the greatest power of x both at the numerator and the denominator we obtain:

$$\lim_{x\to+\infty} \frac{x^2-x-1}{3x^3-2x-3} = \lim_{x\to+\infty} \frac{\cancel{x^2}\left(1-\frac{1}{x}-\frac{1}{x^2}\right)}{x^{\cancel{3}}\left(3-\frac{2}{x}-\frac{3}{2}\right)} = \frac{1-0-0}{+\infty(3-0-0)} = \frac{1}{+\infty} = 0. \qquad \square$$

Exercise 11. *Calculate*

$$\lim_{x\to-\infty} \frac{e^x}{x+2}.$$

Solution. By substituting and remembering the rules of limits we obtain:

$$\lim_{x\to-\infty} \frac{e^x}{x+2} = \frac{e^{-\infty}}{-\infty+2} = \frac{0}{-\infty} = 0. \qquad \square$$

Exercise 12. *Calculate*

$$\lim_{x\to-\infty} \frac{x+2}{e^x}.$$

Solution. By substituting and remembering the rules of limits we obtain:

$$\lim_{x\to-\infty} \frac{x+2}{e^x} = \frac{-\infty+2}{e^{-\infty}} = \frac{-\infty}{0^+} = -\infty.$$

Observe that in this exercise, unlike the previous, the "sign of 0" is important to conclude that the limit is $-\infty$ (and not $+\infty$). $\square$

6 A brief outline on the strength of infinites

Assume that f and g are two functions with the following property:

$$\lim_{x\to\pm\infty} f(x) = \pm\infty, \quad \text{and} \quad \lim_{x\to\pm\infty} g(x) = \pm\infty.$$

If

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \pm\infty,$$

we say that f overhelms g as $x \to \pm\infty$, or that f is an infinite of order greater than g, or again that f is stronger than g: we write $f \succ g$ or $g \prec f$. If instead

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = 0,$$

we say that g overhelms f as $x \to \pm\infty$, or that g is an infinite of order greater than f, or again that g is stronger than f: we write $g \succ f$ or $f \prec g$.

As regards our course, the following hierarchy of infinites holds:

$$\ln x \prec x^{1/3} = \sqrt[3]{f} \prec x^{1/2} = \sqrt{f} \prec x \prec x^2 \prec \cdots \prec x^{10} \prec \cdots \prec e^x \prec e^{x^2} \prec \ldots$$

This may be useful if you need to quickly calculate the limit where a sum or difference of infinites is involved: *in a sum of infinites, as regards a limit you can ignore the weaker infinites.*

Example 8. *To calculate the limit*

$$\lim_{x\to+\infty} x^3 - x^2 - e^x + \ln x,$$

you can ignore x^3, x^2 *and* $\ln x$. *You have*

$$\lim_{x\to+\infty} x^3 - x^2 - e^x + \ln x = \lim_{x\to+\infty} -e^x = -\infty.$$

The calculation of this limit is not possible with the other techniques we have studied.

Example 9. *To calculate the limit*

$$\lim_{x\to+\infty} \frac{e^{x^2} - e^x + x^3 - \ln x}{e^x + x^{100}}$$

you can proceed as follows:

$$\lim_{x\to+\infty} \frac{e^{x^2} - e^x + x^3 - \ln x}{e^x + x^{100}} = \lim_{x\to+\infty} \frac{e^{x^2}}{e^x} = +\infty.$$

In applications (also in economic ones) this is useful if you need a quick estimation of the asymptotic behaviour of a function.