Tip: Magnitude comparison

Luciano Battaia

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It is well known that the natural exponential function grows much more quickly than positive powers of the independent variable and, conversely, that the natural logarithm grows much more slowly, always compared with positive powers of the independent variable.

It is very useful to make explicit calculations in order to exactly understand what this means. The best way to compare two quantities, say $f(x)$ and $g(x)$, that we suppose positive, is to consider the quotient between them:

— if $\dfrac{f(x)}{g(x)} \approx 1$, f and g are substantially equal;

— if $\dfrac{f(x)}{g(x)} \gg 1$, f is substantially bigger than g, and the larger the ratio, the more f is big with respect to g;

— if $\dfrac{f(x)}{g(x)} \ll 1$, f is substantially smaller than g and the smaller the ratio, the more f is small with respect to g.

The following table illustrates the comparison between the natural exponential function and x, x^{100}, x^{1000}, as x gets greater and greater.

x	$\dfrac{e^x}{x}$	$\dfrac{e^x}{x^{100}}$	$\dfrac{e^x}{x^{1000}}$
1	2.72	2.72	2.72
10	$2.20 \cdot 10^{3}$	$2.20 \cdot 10^{-96}$	$2.20 \cdot 10^{-996}$
100	$2.69 \cdot 10^{41}$	$2.69 \cdot 10^{-157}$	$2.69 \cdot 10^{-1957}$
1000	$1.97 \cdot 10^{431}$	$1.97 \cdot 10^{134}$	$1.97 \cdot 10^{-2566}$
10000	$8.81 \cdot 10^{4338}$	$8.81 \cdot 10^{3942}$	$8.81 \cdot 10^{342}$
100000	$2.81 \cdot 10^{43424}$	$2.81 \cdot 10^{42929}$	$2.81 \cdot 10^{38429}$
1000000	$3.03 \cdot 10^{434288}$	$3.03 \cdot 10^{433694}$	$3.03 \cdot 10^{428294}$
10000000	$6.59 \cdot 10^{4342937}$	$6.59 \cdot 10^{4342244}$	$6.59 \cdot 10^{4335944}$
100000000	$1.55 \cdot 10^{43429440}$	$1.55 \cdot 10^{43428648}$	$1.55 \cdot 10^{43421448}$

Observe that even in the case of x^{1000}, e^x overhelms the power, provided that the value of x is large enough. For example if $x = 10000$, x^{1000} is an enormously large number, precisely 1 followed by 40000 zeros, but e^x is much greater than this, in fact it is approximately 1 followed by 40343 zeros!