Università Ca' Foscari di Venezia - Dipartimento di Economia - A.A.2016-2017

Mathematics (Curriculum Economics, Markets and Finance)

One variable optimization

Luciano Battaia

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These notes are a summary of chapter 8 of the textbook in use: see that chapter for further details and explanations.

In all that follows f is a function with domain $D \subseteq \mathbb{R}$ and codomain $\mathbb{R}$.

1 Definitions

— A point $c \in D$ is a *maximum point* for f if and only if $f(x) \leq f(c)$ for *all* $x \in D$.

— A point $d \in D$ is a *minimum point* for f if and only if $f(x) \geq f(c)$ for *all* $x \in D$.

Sometimes these points are also called *global maximum points* or *global minimum points*, and this refers to the fact the previous inequalities hold for *all* point belonging to D. However maximum point or global maximum point are synonyms, and the same for minimum points.

— If c is a maximum point, $f(c)$ is called the *maximum value* of f in D.

— If d is a minimum point, $f(d)$ is called the *minimum value* of f in D.

Observe that *many* maximum points may exist, but at least *one* maximum value exists. The same for the minimum.

If the previous inequalities are replaced by strict inequalities (that is $f(x) < f(c)$ or $f(x) > f(d)$ respectively), then c is a *strict* maximum point, and d a *strict* minimum point.

We can also use collective names such *optimal points* or *extreme points* if we do not need to bother about the distinction between maxima and minima. The same for the corresponding values.

If the previous inequalities hold *only* for all x belonging to an appropriate interval surrounding the points c or d respectively, and not for all x in D, the points are called *local maximum points*, or *local minimum points*, and the corresponding values *local maximum values* or *local minimum values*. In contrast with global maximum or minimum values, many local maximum or minimum values may exist.

A function may also have no local or global maximum or minimum point. Figures 1 to 9 illustrate some possibilities. In all these examples the domain of the function is $\mathbb{R}$: the drawing only reproduces the central part of the graph.

In some cases to find maximum or minimum points one can use only elementary techniques or his knowledge of elementary functions, but in general this is not possible and the location of extreme points may be not elementary: we'll show in the next sections how to proceed at least in the simple cases that are of interest for our course.

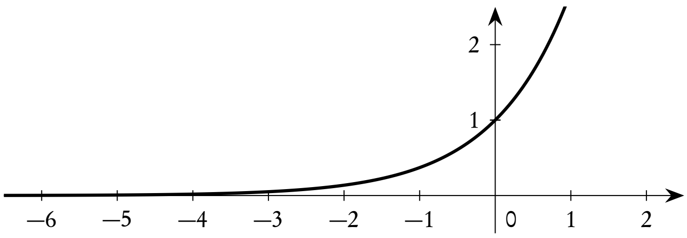

Figure 1 *The natural exponential function has neither maximum nor minimum points*

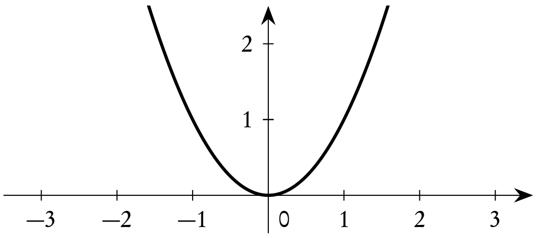

Figure 2 *The parabola* $f(x) = x^2$ *with a minimum point and no maximum points*

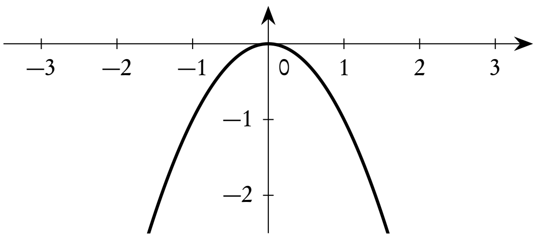

Figure 3 *The parabola* $f(x) = -x^2$ *with a maximum point and no minimum points*

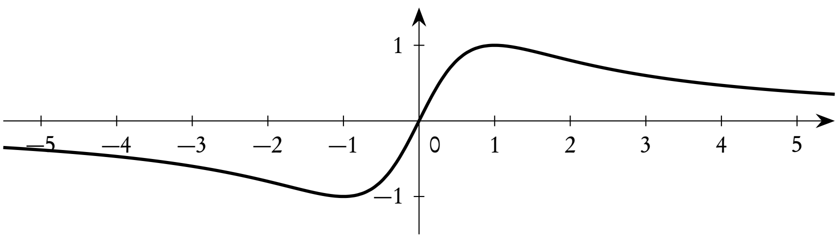

Figure 4 *The function* $f(x) = 2x/(x^2 + 1)$ *with a maximum and a minimum point*

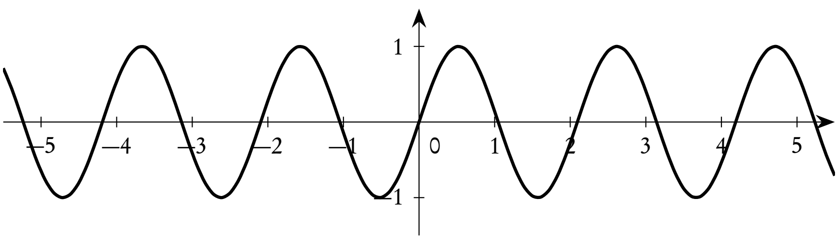

Figure 5 *The function* $f(x) = \sin(3x)$ *with infinitely many maximum and minimum points*

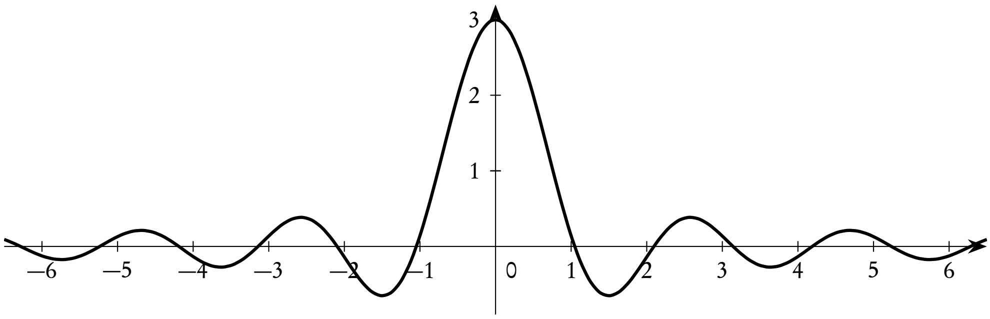

Figure 6 *The function $f(x) = (\sin(3x))/x$ with a maximum point, two minimum points and infinitely many local maximum and minimum points*

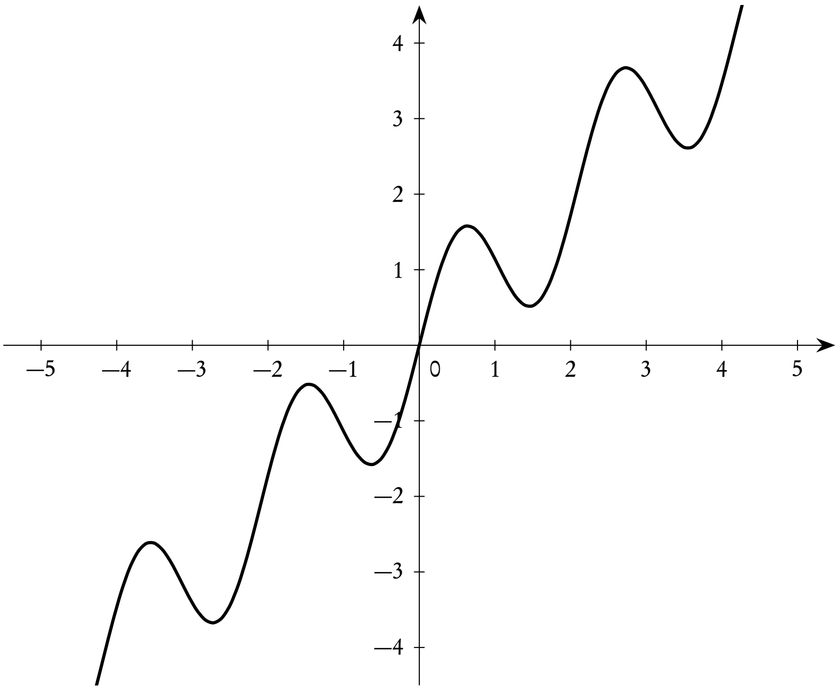

Figure 7 *The function $f(x) = x + \sin(3x)$ with infinitely many local maximum and minimum points, but no global maximum or minimum point*

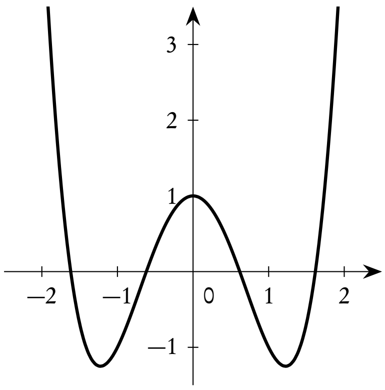

Figure 8 *The function $f(x) = x^4 - 3x^2 + 1$ with two minimum points, one local maximum point, but no maximum point*

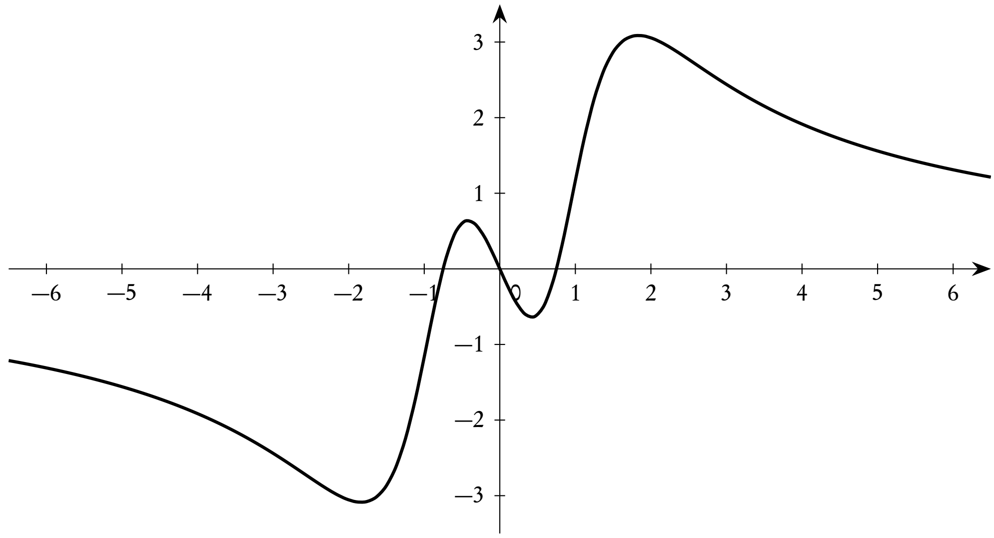

Figure 9 *The function $f(x)=(16x^3-9x)/(2x^4+2)$ with a maximum and a minimum point and also a second local maximum and a second local minimum point*

2 Locating extreme points only by means of the first derivative

The most important technique to locate extreme points is to find the sign of the first derivative: remember that a function is increasing or decreasing right depending on the sign of the first derivative.

We are only interested in functions that are almost everywhere continuous, except perhaps a limited number of points, usually one or two at the most; furthermore our functions are also usually everywhere differentiable, except perhaps a limited number of points, also one or two at the most.

If a function is everywhere differentiable (and consequently everywhere continuous), proceed as follows:

1. compute the limits at the boundaries of the domain, or the values of the function at these boundaries, if they are included in the domain;
2. compute the first derivative;
3. find where the derivative is positive or negative; remember that when the derivative is zero the tangent is horizontal, but this does not imply that this is an extreme;
4. deduce the appropriate consequences.

If there is some point where the function is not continuous or not differentiable, carefully examine the behaviour of the function near this point: it will not be difficult to deduce the appropriate consequences.

Let's consider some example in order to clarify how to proceed.

Example 1. *Given the function*

$$f(x)=2x^3-9x^2+12x-3,$$

find its local extremum points and, if they exist, its maximum and minimum value.

The natural domain of the function is $\mathbb{R}$ and the limits at the boundaries are

$$\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty} x^3\left(2-\frac{9}{x}+\frac{12}{x^2}-\frac{3}{x^3}\right)=\pm\infty.$$

This function can't have a maximum or minimum value.

The first derivative is

$$f'(x)=6x^2-18x+12.$$

This derivative is positive in $]-\infty, 1[$ and in $]2, +\infty[$, while it is negative in $]1, 2[$. We conclude that there is a local maximum at $x = 1$ and a local minimum at $x = 2$.

A sketch of the graph is plotted in figure 10, but this graph is not important to draw the conclusions.

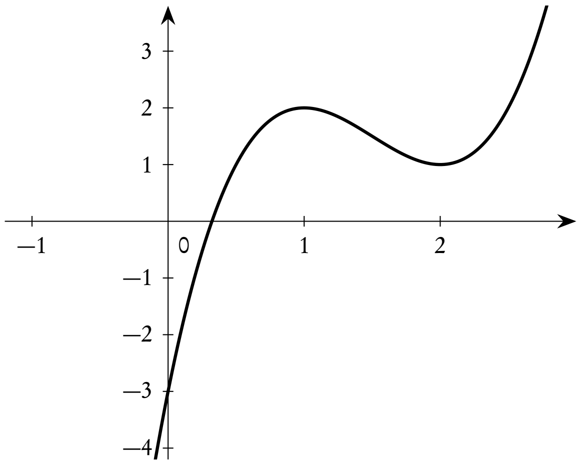

Figure 10 *Plot of the graph of example 1*

Example 2. *Consider again the function in example 1, but in the interval* $[0, 2]$, *and find the maximum and minimum.*

Now the maximum and minimum surely exist as a consequence of the extreme value theorem (Weierstrass' theorem). In addition to previous calculations we need only to compute

$$f(0) = -3, \quad f(1) = 2 \quad \text{and} \quad f(2) = 1.$$

We can conclude that function has -3 as minimum value (attained at $x = 0$) and 2 as maximum value (attained at $x = 1$.

Example 3. *Given the function*

$$f(x) = \begin{cases} x^2, & \text{if } x < 1; \\ 2, & \text{if } x = 1; \\ x - 1, & \text{if } x > 1; \end{cases},$$

find its extreme local and global values, if they exist.

No need to make derivatives: this compound function is defined in terms of known functions, precisely a parabola and a straight line. It's easy to conclude that the function has no maximum, a relative maximum point at $x = 1$, with value 2, a minimum point at $x = 0$, with value 0. Pay attention: there is no minimum point at $x = 1$, even if the limits as x tends to 1 from above is 0. The graph is plotted in figure 11.

Example 4. *Consider again the function of example 3, but in the interval* $[-1, 2]$, *and find the maximum and minimum.*

Now the maximum and minimum surely exist as a consequence of the extreme value theorem (Weierstrass' theorem). In addition to previous calculations we need only to compute

$$f(-1) = 1 \quad \text{and} \quad f(2) = 1.$$

The maximum of the function is 2, attained at $x = 1$, while the minimum is 0, attained at $x = 0$.

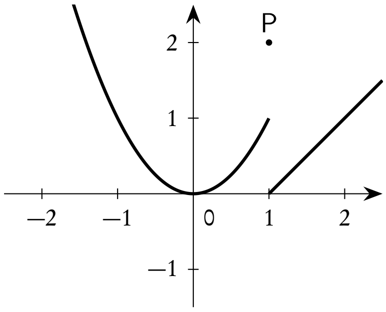

Figure 11 *Plot of the graph of example 3*

Example 5. *Consider the function*

$$f(x) = \frac{1}{x^2}, \quad 0 < x \le 2,$$

and compute, if they exist, its maximum and minimum value.

The limits as $x \to 0^+$ is $+\infty$, so the function can't have a maximum. The first derivative is

$$f'(x) = -\frac{2}{x^3}.$$

For $x > 0$ this derivative is always negative, so the function is decreasing in the given domain. As

$$f(2) = \frac{1}{4},$$

the minimum is precisely $1/4$.

The graph is plotted in figure 12.

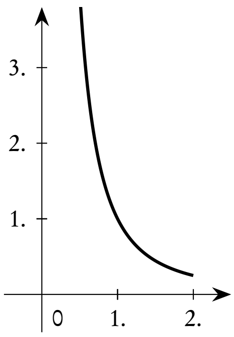

Figure 12 *Plot of the graph of example 5*

3 Use of second order derivatives

If a function is twice differentiable it is possible to locate maxima and minima with the help of the second derivative: if at a given point the first derivative is zero and the second derivative is positive, that point is a local minimum point, if the second derivative is negative, that point is a local maximum point.

This technique is useful in that it makes not necessary to calculate the sign of the first derivative. However, in the cases of our interest, the sign of the first derivative is always easy enough and we suggest you to always find the signs of the first derivative and to conclude accordingly.

However it is very important to point out the fact that, when considering functions of many variables no technique only based on the first derivative exists: in these case we are forced to compute second derivatives and to conclude keeping in mind something that is strictly connected with second derivatives.