Linear dependence or independence

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These notes are a supplement to the book *Essential Mathematics for Economic Analysis* of K.Sydsæter, P.Hammond, A.Strøm & Andrès Carvajal.

Let's begin by recalling the concept of *vector* of $\mathbb{R}^n$. A matrix with only one row and n columns, $A_{1\times n}$, is called a *row vector*, while a matrix with n rows and one column, $A_{n\times 1}$ is called a *column vector*, or simply a *vector*. If there is no specification, a vector is always a column vector. Vectors are represented with a lowercase letter topped by an arrow: $\vec{v}$. Also bold lowercase letters are used.

Definition 1. *Given any number of vectors of* $\mathbb{R}^n$, $\vec{v}_1$, $\vec{v}_2$, $\dots$, $\vec{v}_r$, *the vector* $\vec{w}$

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r,$$

where c_1, c_2, $\dots$, c_r *are real numbers is* a linear combination *of the vectors* $\vec{v}_1$, $\vec{v}_2$, $\dots$, $\vec{v}_r$.

The numbers c_1, c_2, $\dots$, c_r *are the coefficients of the combination.*

Example 1. *Given the row vectors* $\vec{v}_1 = (1,1,0)$, $\vec{v}_2 = (0,0,1)$ *it is easy to verify that* $\vec{v}_3 = (3,3,-2)$ *is a linear combination of* $\vec{v}_1$ *and* $\vec{v}_2$. *In fact we have*

$$\vec{v}_3 = 3\vec{v}_1 - 2\vec{v}_2,$$

Example 2. *Given the row vectors* $\vec{v}_1 = (1,0,0)$, $\vec{v}_2 = (0,1,0)$ *and* $\vec{v}_3 = (0,0,1)$ *it is easy to verify that no one of them can be a linear combination of the other two: for example any combination of the vectors* $\vec{v}_2$ *and* $\vec{v}_3$ *will have a* 0 *as first entry and so will not be possible to obtain the vector* $\vec{v}_1$.

The last example is particularly important for the subject of these notes. The following is in fact the main concept we are interested in.

Definition 2. *The vectors* $\vec{v}_1$, $\vec{v}_2$, $\dots$, $\vec{v}_r$ *of* $\mathbb{R}^n$ *are* linearly dependent *if at least one of them is a linear combination of the others,* linearly independent *if no vector can be written as a linear combination of the others.*

The question of whether the vectors of a given set are linearly dependent or not leads to the resolution of a linear system, as shown in the following example.

Example 3. *Let's consider the row vectors* $\vec{v}_1 = (1,-1,2)$, $\vec{v}_2 = (2,1,3)$ *e* $\vec{v}_3 = (1,0,3)$ *and suppose we want to check whether* $\vec{v}_3$ *is a linear combination of* $\vec{v}_1$ *and*$\vec{v}_2$. *We must solve the following equation:*

$$c_1\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

with c_1 and c_2 as unknowns. This condition can be written as

$$\begin{cases} c_1 & + & 2c_2 & = & 1 \\ -c_1 & + & c_2 & = & 0 \\ 2c_1 & + & 3c_2 & = & 3 \end{cases}.$$

It is easy to prove that this system has no solution: $\vec{v}_3$ is not a linear combination of the other two. In a similar way it can be proved that $\vec{v}_1$ cannot be written as a combination of $\vec{v}_2$ and $\vec{v}_3$ and, similarly, that $\vec{v}_2$ cannot be written as a combination of $\vec{v}_1$ and $\vec{v}_3$. So the three vectos are linearly independent.

As an immediate consequence of the definition we can obtain the following theorem.

Theorem 3. *The vectors $\vec{v}_1$, $\vec{v}_2$, …, $\vec{v}_r$ of $\mathbb{R}^n$ are linearly independent if and only if from the equality*

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r = \vec{0} = (0, 0, \ldots, 0)$$

it follows that $c_1 = c_2 = \cdots = c_r = 0$. If, on the contrary, the previous equality holds with at least one of the coefficients different from 0, then the vectors are linearly dependent.

Example 4. *Check if the following vectors of $\mathbb{R}^3$ are dependent or not.*

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix}.$$

Let's consider the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

in the unknowns c_1, c_2, c_3. We can write the same equation as

$$\begin{pmatrix} c_1 + 2c_2 - 4c_3 \\ -3c_1 - c_2 + 2c_3 \\ 7c_1 - c_2 + 2c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

or

$$\begin{cases} c_1 + 2c_2 - 4c_3 = 0 \\ -3c_1 - c_2 + 2c_3 = 0 \\ 7c_1 - c_2 + 2c_3 = 0 \end{cases}$$

Observe that $c_1 = c_2 = c_3 = 0$ is certainly a solution of this system (called on homogeneous system*). The problem is to check if there are also other solutions. With the usual methods we can find that the system has infinitely many solutions; if we choose c_3 as parameter they are:*

$$c_1 = 0, \quad c_2 = 2t, \quad c_3 = t$$

Taking into consideration all what we have studied for linear systems we can state the following theorem, that greatly simplifies the problem of checking if some vectors are dependent or not.

Theorem 4. *The rank of a matrix A is the maximum number of row vectors or column vectors linearly independent.*

As a consequence of this theorem we can conclude that k vectors of $\mathbb{R}^n$ are independent if and only if the matrix obtained by writing the vectors as columns (or rows) has rank k.

So, for example, 4 vectors of $\mathbb{R}^3$ cannot be independent because the matrix with the vectors as columns can have at maximum rank 3 (3 rows).