The theorem of Rouché & Capelli

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These notes are a supplement to the book *Essential Mathematics fo Economic Analysis* of K.Sydsæter, P.Hammond, A.Strøm & Andrès Carvajal.

Let's begin with the introduction of the concept of *minor* for a general matrix.

Given an arbitrary matrix $A_{m\times n}$, square or rectangular, choose k rows and k columns, also non contiguous: the common entries to these k rows and k columns give rise to a square matrix, called a *submatrix* or an *extracted matrix* of the matrix A; the determinant of this extracted matrix is called *minor*, o extracted minor, of order k. You can also say that an extracted matrix is a matrix obtained from the given matrix by erasing some rows and columns, with the condition that the remaining rows and columns make up a square matrix.

For example from the matrix $A_{2\times 3}$

$$A=\begin{pmatrix} 1 & 3 & -1 \\ -2 & 1 & -3 \end{pmatrix},$$

we can extract 6 minors of order 1

$$1, 3, -1, -2, 1, -3,$$

and three minors of order 2

$$\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix}=7, \quad \begin{vmatrix} 1 & -1 \\ -2 & -3 \end{vmatrix}=-5, \quad \begin{vmatrix} 3 & -1 \\ 1 & -3 \end{vmatrix}=-8, \quad .$$

The greatest positive number r for which there exists a minor of order r different form 0 is called the *rank* of the matrix.

Example 1. *The rabk of the matrix*

$$A=\begin{pmatrix} 1 & 4 & 3 & -1 \\ -2 & -1 & 1 & -3 \\ 2 & 1 & -1 & 4 \end{pmatrix}$$

is 3*, because, for example, the minor of order* 3 *obtained considering the first, third and fourth column is* 7*, as you can easily prove:*

$$\begin{vmatrix} 1 & 3 & -1 \\ -2 & 1 & -3 \\ 2 & -1 & 4 \end{vmatrix}=7.$$

Example 2. *The rank of the matrix*

$$A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 7 & -5 & 5 \\ -2 & 1 & -3 & 1 \\ 2 & -8 & 8 & -6 \end{pmatrix},$$

is 2*, because the only minor of order* 4 *(that is the determinant of the matrix itself) is* 0 *(prove this!), all minors of order* 3 *are also* 0 *(prove this as an exercise), while for example the minor of order* 2 *obtained at the intersection of the first* 2 *rows and columns is* 7*.*

To compute the rank of a matrix proceed as follows.

- Consider the extracted matrices of the greatest possible order: as soon as you obtain a determinant of one of these matrices different from 0 stop the computation; you can conclude that the rank is the order of this matrix.
- If all the determinants of the previous extracted matrices are 0, consider all the matrices of one order lower: as soon as you obtain a determinant of one of these matrices different from 0 stop the computation; you can conclude that the rank is the order of this matrix.
- Proceed like that till you obtain a determinant different form 0.

Observe (very very important!!) that the rank of a matrix can be only a natural number (unlike the determinant that can be a general real number) and that the rank is always lower than or equal to the minimum numbers of rows or columns of the given matrix. If, besides that, the matrix is not a null matrix, the rank is at least 1.

Let's now recall some concepts also considered in the textbook.

For a linear system (with any number of equations and unknowns) there are only three possibilities as regards the solutions:

1. the system can have no solution at all: in this case it is called *inconsistent*;
2. the system can have only one solution;
3. the system can have infinitely many solutions.

A system having solutions (one or infinitely many) is called *consistent*.

To check the consistency of a linear system with m equations and n unknowns we must consider the *coefficient matrix*, A, and the *augmented matrix* $A|b$, obtained from A by adding the (column) vector of the constants to the right.

$$A|b = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right). \tag{1}$$

Separating the last column with a vertical bar helps to write in a compact form both the coefficient matrix and the augmented matrix.

The consistency or inconsistency of the systm can be checked by means of the following theorem.

Theorem 1 (Rouché-Capelli' theorem)**.** *A linear systrem of m equations in n unknowns is consistent if and only if the coefficient matrix and the augmented matrix have the same rank, called* the rank of the system.

Example 3. *The linear system with augmented matrix*

$$A|b = \left(\begin{array}{cc|c} 3 & 4 & 7 \\ 5 & 6 & 11 \\ 2 & 7 & 4 \end{array}\right)$$

has no solution, because the augmented matrix has determinant

$$\det(A|b) = \begin{vmatrix} 3 & 4 & 7 \\ 5 & 6 & 11 \\ 2 & 7 & 4 \end{vmatrix} = 10 \neq 0,$$

and consequently rank 3*, while the coefficient matrix can have at most rank* 2*, in that it has only two columns.*

Example 4. *The linear system with augmented matrix*

$$A|b = \left(\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 4 & 5 \\ 1 & 5 & 1 \end{array}\right)$$

is consistent. In fact $\det(A|b) = 0$*, while, for example,*

$$\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$$

and then the two matrices have rank 2*. Observe that the rank of the coefficient matrix is always greater than or equal to the rank of the augmented matrix.*

As the last example shows, to check the consistency of a system we must find a non zero minor of the greatest possible order that is a minor both of the coefficient and the augmented matrices. After this check the resolution od the system can proceed as follows.

1. Erase all the the equations (if any) corresponding to the rows that have not been used for the found minor.
2. Move to the second member all the terms (if any) that contain unknowns in columns not used for the found minor: these unknowns will remain arbitrary and are called *parameters*.
3. In this way you obtain a square system that can be solved by Cramer's rule, or using the inverse matrix strategy, or by substitution.

If the system is consistent and there is no parameter, then it has only one solution. If there are parameters, the system has infinitely many solutions and, if there are k parameters usually we say that the system has ∞^k solutions.

For the sake of completeness we recall Cramer's theorem.

Theorem 2 (Cramer's theorem)**.** *Consider a linear system of n equations in n unknowns, with coefficient matrix A, whose determinant* $\det(A)$ *is different from* 0*. Consider the matrices* A_i*, obtained by replacing the* i*-th column of A with the column* $\vec{b}$*. The unique solution of the system is*

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \dots, \quad x_n = \frac{\det(A_n)}{\det(A)}.$$

Example 5. *The system with augmented matrix*

$$A|b = \left(\begin{array}{ccc|c} 3 & 2 & 2 & 13 \\ 2 & 4 & 3 & 19 \\ 4 & 5 & 2 & 20 \end{array}\right)$$

is consistent because the coefficient matrix has $\det(A) = -17 \neq 0$*, and so rank* 3*. The augmented matrix must have a rank* ≥ 3*, but it has only* 3 *rows, so it has rank* 3*. Let's consider the* 3 *matrices* A_i*:*

$$\det(A_1) = \begin{vmatrix} 13 & 2 & 2 \\ 19 & 4 & 3 \\ 20 & 5 & 2 \end{vmatrix} = -17, \ \det(A_2) = \begin{vmatrix} 3 & 13 & 2 \\ 2 & 19 & 3 \\ 4 & 20 & 2 \end{vmatrix} = -34, \ \det(A_2) = \begin{vmatrix} 3 & 2 & 13 \\ 2 & 4 & 19 \\ 4 & 5 & 20 \end{vmatrix} = -51.$$

The solution is then

$$x_1 = \frac{-17}{-17} = 1, \quad x_2 = \frac{-34}{-17} = 2, \quad x_1 = \frac{-51}{-17} = 3,$$

that can be written as a column vector:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Example 6. *The system with augmented matrix*

$$A|b = \left(\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 4 & 3 & -1 & 2 \\ -2 & -1 & 1 & 0 \end{array}\right)$$

has rank 2. *In fact the* 4 *minors of order* 3 *of the augmented matrix are* 0 *(and consequently also the determinant of the coefficient matrix itself)*

$$\begin{vmatrix} 3 & 2 & -1 \\ 4 & 3 & -1 \\ -2 & -1 & 1 \end{vmatrix} = 0, \begin{vmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ -2 & -1 & 0 \end{vmatrix} = 0, \begin{vmatrix} 3 & -1 & 1 \\ 4 & -1 & 2 \\ -2 & 1 & 0 \end{vmatrix} = 0, \begin{vmatrix} 2 & -1 & 1 \\ 3 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = 0,$$

while the submatrix obtained by taking the intersection of first two rows and columns has determinant $9 - 8 = 1 \neq 0$.

If we erase the third equation and move to the second member all the terms containing the third unknown, we obtain

$$\left(\begin{array}{cc|c} 3 & 2 & 1 + x_3 \\ 4 & 3 & 2 + x_3 \end{array}\right).$$

Using Cramer's rule we obtain

$$x_1 = \frac{\begin{vmatrix} 1 + x_3 & 2 \\ 2 + x_3 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}} = x_3 - 1, \qquad x_2 = \frac{\begin{vmatrix} 3 & 1 + x_3 \\ 4 & 2 + x_3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}} 2 - x_3,$$

while x_3 *remains arbitrary: let's call it* t. *The solutions are*

$$\begin{pmatrix} t - 1 \\ 2 - t \\ t \end{pmatrix},$$

that is ∞^1 *solutions.*

Example 7. *A factory manufactures* 3 *products,* P_1, P_2, P_3. *Each product undergoes a working cycle in* 3 *different departments, A, B, C, with the following working times, in hours:*

	A	B	C
P_1	2	1	1
P_2	5	3	2
P_3	3	2	2

If for a specific production the 3 *departments have worked for* 104 *hours,* 64 *hours and* 55 *hours respectively for the departments A, B and C, we ask how many pieces of each product have been produced.*

If the quantities we are looking for are x_1, x_2, x_3 *we must have:*

$$\begin{cases} 2x_1+5x_2+3x_3=104 \\ x_1+3x_2+2x_3=64 \\ x_1+2x_2+2x_3=55 \end{cases}.$$

We need to solve a linear system with augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 5 & 3 & 104 \\ 1 & 3 & 2 & 64 \\ 1 & 2 & 2 & 55 \end{array}\right).$$

As the determinant of the coefficient matrix is 1*, the system can be solved using Cramer's rule and we obtain:*

$$x_1=7, \quad x_2=9, \quad x_3=15.$$

Let's use also the inverse matrix strategy. If

$$A=\begin{pmatrix} 2 & 5 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}, \quad \vec{b}=\begin{pmatrix} 104 \\ 64 \\ 55 \end{pmatrix}, \quad \vec{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

the system can be written in compact form.

$$A\vec{x}=\vec{b}.$$

The inverse of the matrix A *is*

$$A^{-1}=\begin{pmatrix} 2 & -4 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

We can now compute the solution as follows

$$\vec{x}=A^{-1}\vec{b}=\begin{pmatrix} 2 & -4 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 104 \\ 64 \\ 55 \end{pmatrix}=\begin{pmatrix} 7 \\ 9 \\ 15 \end{pmatrix},$$

with the same result.

Observe that this strategy is exactly the same we normally use to solve a first degree equation in one unknown:

$$ax=b, \quad a\neq 0, \quad \Rightarrow \quad x=a^{-1}b.$$