Basics of Financial Mathematics

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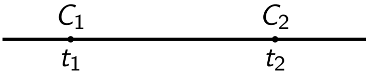
October 19, 2017

The Fundamental Problem of Financial Mathematics

If you have two capitals, or amounts of money, C_1 and C_2, at the times t_1 and t_2, the FP is to establish a rational comparison criterium between the two in order to decide whether they are equally valuable or which is more valuable or preferable, with the obvious assumption that if $t_1 = t_2$ than the greatest one is preferable.



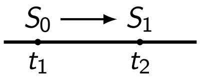
Suppose you have at a given time (initial time) a capital S_0, called Initial Capital, or Principal, or Present Value, PV.
If you deposit it in a bank, at the end of a unit period of time (usually a year), you'll have S_0 plus a portion of S_0, called the Interest, I. So the final amount will be $S_1 = S_0 + I$. This is also called the Final Capital, F, or Future Value, FV.
The quotient $r = I/S_0$ is called the interest rate or nominal interest rate and is (unless it is a usury rate!!) less than 1.

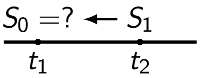
Example. $S_0 = 1000$, $I = 100$, $r = 100/1000 = 0.1$

Usually r is expressed as a percentage:
$0.1 = 0.10 = 10/100 = 10\%$. In general $r = p/100 = p\%$.

So: $S_1 = S_0 + rS_0 = S_0(1 + r) = fS_0$ and $f = 1 + r$ is called the accumulation factor.

The inverse problem, even if it is exactly equivalent, is interesting and deserves special names.





$$S_0 = \frac{S_1}{1+r} = S_1 d, \quad \text{where} \quad d = \frac{1}{1+r}.$$

The number d is called the discount factor or actualization factor.

Example. (Here and in all future examples we'll take the year as time unit, so "r" is called the annual rate of interset).

If the annual interest rate is 9% ($r = 0.09$), find if it is preferable to have 120 today or 125 after one year.

1. Compare the present values:

   The present value of 125 is $\frac{125}{1+r} = \frac{125}{1.09} \approx 115$.

2. Compare the future values:

   The future value of 120 is $120(1 + r) = 120 \cdot 1.09 \approx 131$.

We can conclude that it is preferable to have 120 today than 125 after one year.

Observe that, despite the simplicity of this example, it highlights exactly the very fundamental problem of financial mathematics.

Example. If 120 is the FV of 90 in 1 year, what is r?

$$120 = 90(1 + r) \quad \Rightarrow \quad 30 = 90r \quad \Rightarrow \quad r = \frac{1}{3} = 0.33 = 33\%.$$

Example. If 120 is the PV of 150 in 1 year, what is the interest rate? What is the discount rate?

$$150 = 120(1 + r) \quad \Rightarrow \quad 30 = 120r \quad \Rightarrow$$

$$r = 0.25 = 25\% \quad \Rightarrow \quad d = \frac{1}{1 + r} = 0.8 = 80\%.$$

The usual financial regime is that of “compound interest”, that is after the first period (year) the interest is added to the principal and it contributes to produce the new interest for the second year. So:

$$\text{Initial value: } S_0\,, \quad \text{after 1 year: } S_1 = S_0(1+r)\,,$$

$$\text{after 2 years: } S_2 = S_1(1+r) = S_0(1+r)^2.$$

In general, after t years, the Final Capital will be

$$S_t = S_0(1+r)^t.$$

Now suppose that the interest is added to the principal not at the end of the year, but *biannually*, that is after six months (so it contributes to produce new interest) and finally at the end of the year.

After half a year you have the following amount

$$S_0 + r\frac{S_0}{2} = S_0\left(1 + \frac{r}{2}\right).$$

So at the end of the year you'll have

$$S_1 = S_0\left(1 + \frac{r}{2}\right)^2.$$

If the year is divided into n periods the formula obviously becomes, respectively for one year and for t years,

$$S_1 = S_0\left(1 + \frac{r}{n}\right)^n \quad \text{and} \quad S_t = S_0\left(1 + \frac{r}{n}\right)^{nt}.$$

When the period is divided in sub-periods the effective rate of interest I/S_0 for the whole period is obviously greater than the nominal rate of interest. This rate of interest is called Effective rate of interest and usually denoted with R.

$$R = \frac{I}{S_0} = \frac{S_1 - S_0}{S_0} = \frac{S_0 \left(1 + \frac{r}{n}\right)^n - S_0}{S_0} = \left(1 + \frac{r}{n}\right)^n - 1.$$

Example. If the principal is 1000 and the nominal rate of interest is 9%, in one year the FV would be 1090. If the interest, at the same nominal rate, is added monthly, that is 12 times a year, the FV, after one year, will be:

$$1000 \left(1 + \frac{0.09}{12}\right)^{12} \approx 1094,$$

and it is as if the rate of interest where 9.4% instead of 9%. In fact

$$R = \left(1 + \frac{0.09}{12}\right)^1 2 - 1 \approx 0.094 = 9.4\%.$$

Continuous compounding

Suppose, in the formula

$$S_t = S_0 \left(1 + \frac{r}{n}\right)^{nt} \ ,$$

that $n \to +\infty$: in this case we'll speak of continuous compounding. We obtain

$$S_t = S_0 \left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = S_0 \left[\left(1 + \frac{1}{n/r}\right)^{n/r}\right]^{rt} = S_0 \left[\left(1 + \frac{1}{x}\right)^{x}\right]^{rt},$$

where $x = n/r$. As $n \to +\infty$ also $x \to +\infty$ and the function under square brackets tends to the Euler number "e" (this is a fundamental limit). So

$$S_t = S_0 e^{rt}.$$

This is where the "e" number comes from!!

Example. A deposit of 5000 is put into an account earning interest at the annual rate of 9%. How much will there be in the account after 8 years if the interest is paid annually, quarterly or continuously?

$$\text{Annually: } FV = 5000(1 + 0.09)^8 \approx 9963.$$

$$\text{Quarterly: } FV = 5000(1 + 0.09/4)^{32} \approx 10191.$$

$$\text{Continuously: } FV = 5000e^{0.09 \cdot 8} \approx 10272.$$

Stream of Cash Flow

A Stream of Cash Flow is a sequence of payments/incomes, a_0, a_1, a_2, ..., a_n, called *instalments* (or *installments* in US english), at certain times t_0, t_1, t_2, ..., t_n, called *maturities*.

Example. Suppose that three successive payments are to be made in the amount of 1000, after 1 year, 1500, after 2 years and 2000, after 3 years. How much must be deposited in an account today in order to have enough money to cover these three payments if the annual rate is 11%?

This amount is called the Total Present Value of the payments and can be simply computed using actualization factors for the three payments and then summing up:

$$TPV = \frac{1000}{(1+0.11)^1} + \frac{1500}{(1+0.11)^2} + \frac{2000}{(1+0.11)^3} \approx 3581.$$

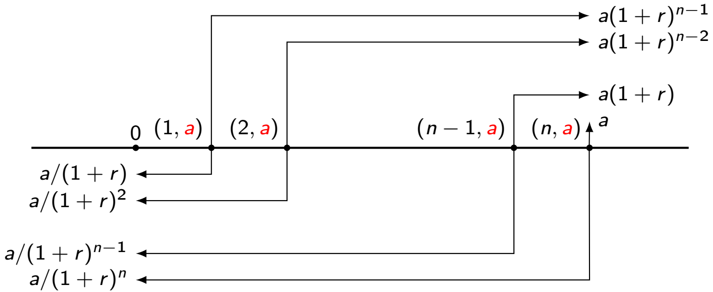
The same pattern can be used to find the Future Value (Total Future Value), using accumulation factors.

We are only interested in the following situation: the period between two consecutive maturities is always the same (a year for example) and the amount of the instalments is a constant value, say a. In this case the SCF is called a (simple) annuity.
The annuity is ordinary if payments are the end of each period, is due, if payments are at the beginning of each period.

The sums in these cases are easily computed if we use the formula for the sum of a geometric progression:

$$1 + q + q^2 + q^3 + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}\,.$$

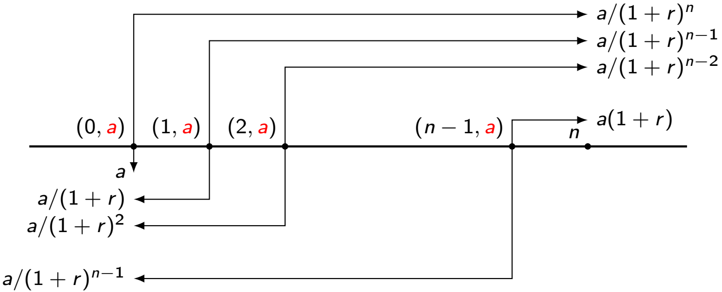
Ordinary annuity



$$TFV = a\left[1 + (1+r) + \cdots + (1+r)^{n-1}\right] = \frac{a}{r}\left[(1+r)^n - 1\right].$$

$$TPV = \frac{a}{1+r}\left[1 + \frac{1}{1+r} + \cdots + \frac{1}{(1+r)^{n-1}}\right] = \frac{a}{r}\left[1 - \frac{1}{(1+r)^n}\right].$$

Due annuity



Exercise. Prove the following.

$$TPV = \frac{a(1+r)}{r}\left[1 - \frac{1}{(1+r)^n}\right], \quad TFV = \frac{a(1+r)}{r}\left[(1+r)^n - 1\right].$$