Università Ca' Foscari di Venezia - Dipartimento di Management - A.A.2017-2018

Mathematics

Lagrangian Multipliers

Luciano Battaia

November 15, 2017

1 Two variables functions and constraints

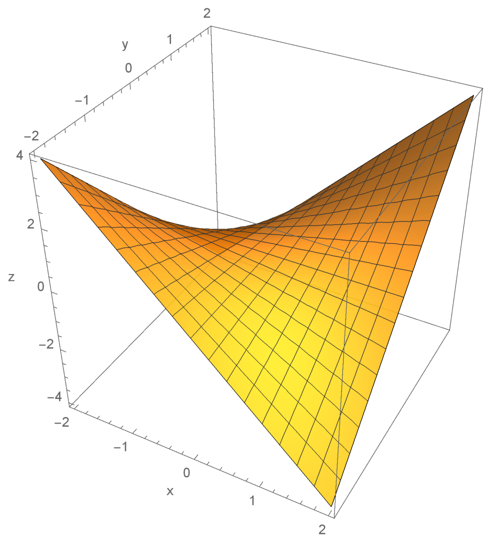
Consider a two variables function $f(x,y)$, with domain $D \subseteq \mathbb{R}^2$. Usually D is a two dimensional region of the cartesian plane, and in many situations it is the entire plane. The problem of finding local extrema at the interior of D can be solved, in the cases of our interest, using the first derivatives and the Hessian.

Now let's consider an equation in two variables, $g(x,y) = 0$[(1)]: usually the set of solutions of this equation is a curve in the cartesian plane. In many applications (also in economic ones) it is of great importance to maximize or minimize the function f not in the entire domain D, but only considering the part of the graph corresponding to points of the curve given by $g(x,y) = 0$. Let's illustrate this fact with a simple example and then a more complicated one.

Example 1. *The function*

$$f(x,y) = xy$$

has the following graph:

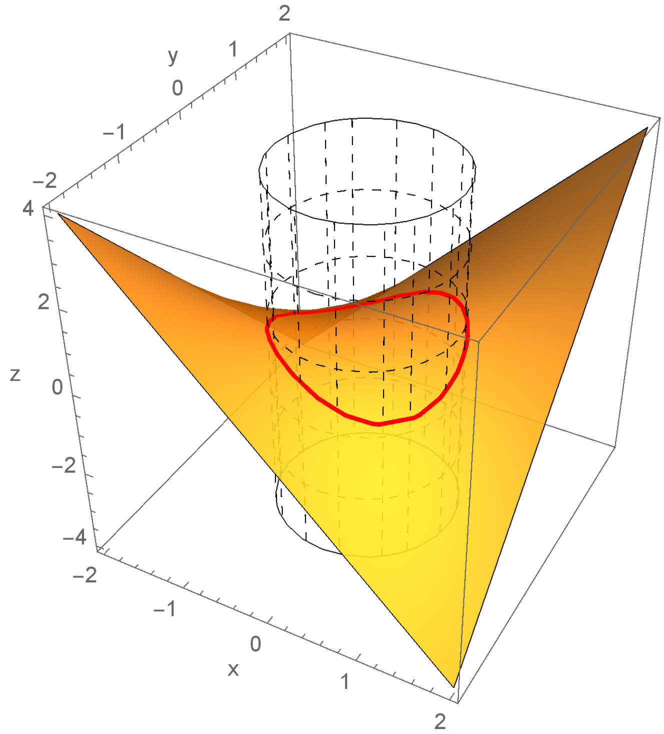


[1]The equation can also be written in the form $h(x,y) = c$. In this case put $g(x,y) = h(x,y) - c$ and you obtain the form $g(x,y) = 0$.

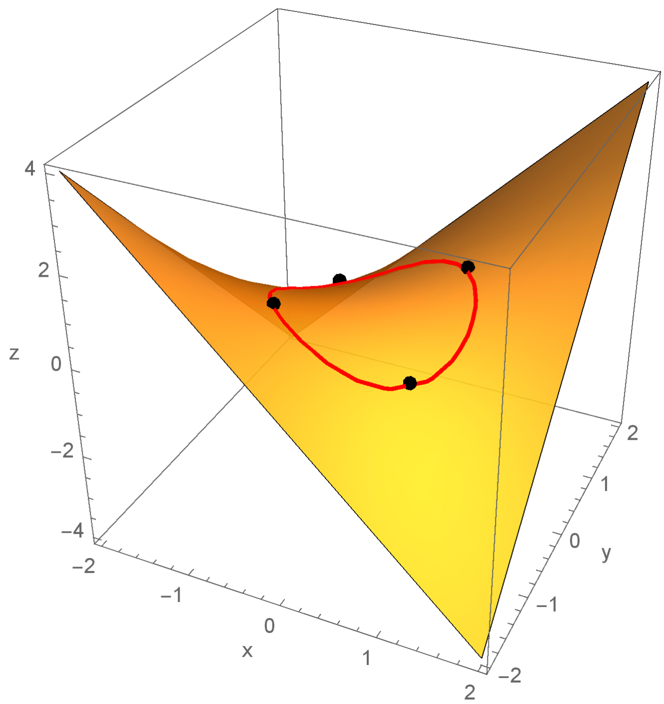
The part of the graph corresponding to the points of the curve

$$x^2 + y^2 - 1 = 0$$

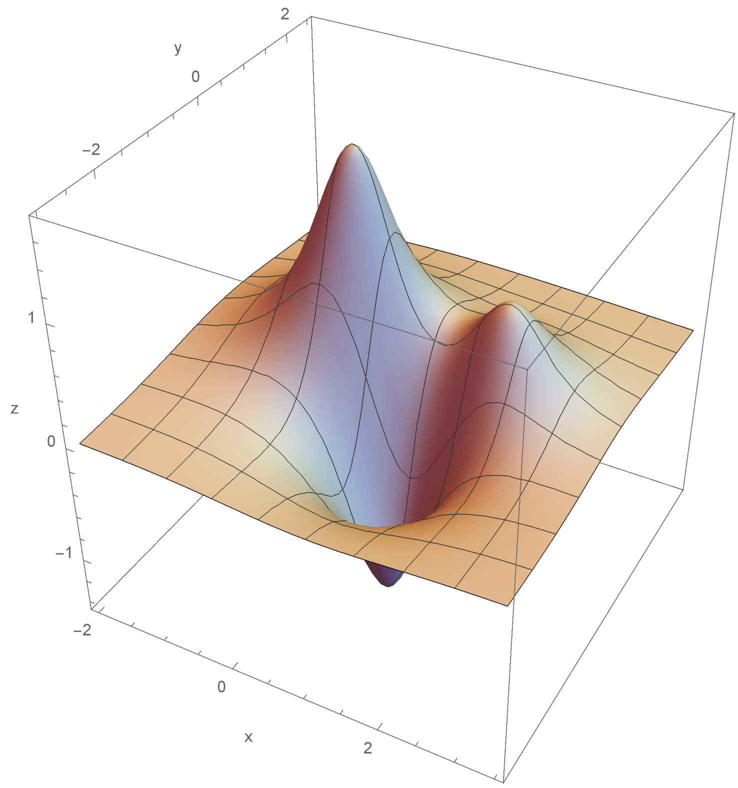
is represented by the red line drawn on the surface graph of f*:*



The problem that we want to treat is to find the global maximum and minimum of this part of the graph. In the following picture you can see the four points where the maximum or minimum are reached.



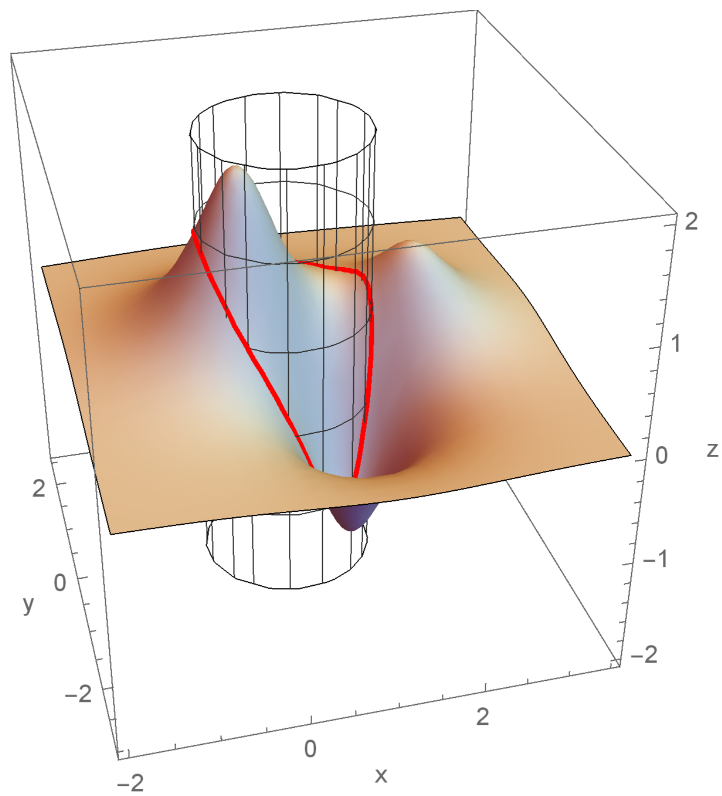
Example 2. *Consider the following graph of a function of two variables, where you can find a global maximum, a local maximum and a global minimum:*



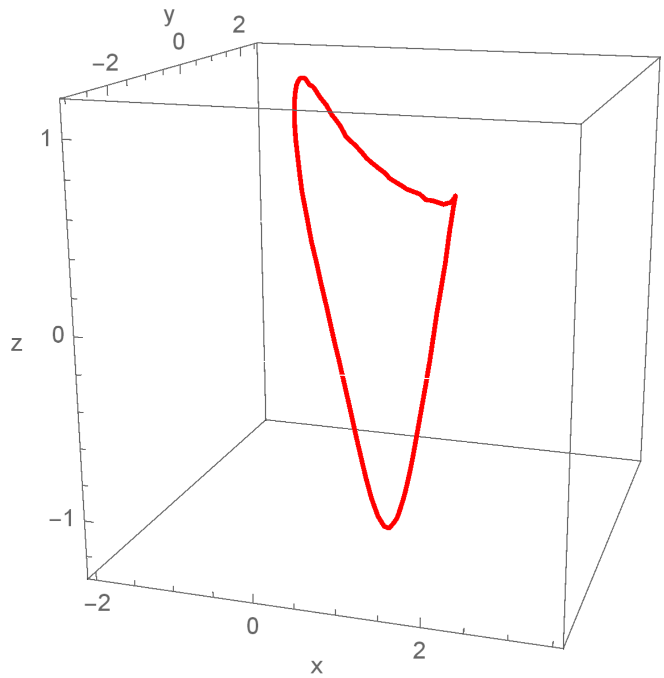
The part of the graph corresponding to the points of the curve

$$(x-1/2)^2+y^2-1=0$$

is represented by the red line drawn on the surface graph of f:



For a better comprehension the red curve is plotted alone in the following graph:



The problem that we want to treat is again to find the global maximum and minimum of this part of the graph.

The curve $g(x,y)=0$ is called a *constraint* and the problem we are dealing with is called a problem of *constrained optimization*.

2 Lagrangian multipliers

If one can use the equation of the constraint to obtain a one variable function from the two variables function f, it is possible to solve the problem as with functions of one variable. But, unfortunately, this is not always the case. If this happens the method of *Lagrangian multiplier* can be used. Proceed as indicated in the following steps.

1. Write down a new function of three variables, calle the *Lagrangian function*, as follows:

$$\mathcal{L}(x,y,\lambda)=f(x,y,)-\lambda g(x,y),$$

   where λ is a supplementary variable, called a *lagrangian multiplier*.
2. Solve the system of three equations in three unknowns:

$$\begin{cases} \mathcal{L}'_x(x,y,\lambda)=f'_x(x,y)-\lambda g'_x(x,y)=0 \\ \mathcal{L}'_y(x,y,\lambda)=f'_y(x,y)-\lambda g'_y(x,y)=0 \\ g(x,y)=0 \end{cases}$$

3. Let (x_1,y_1,λ_1), (x_2,y_2,λ_2), $\ldots$, be the solutions of this system. If the maximum-minimum problem has a solution this solution can only be one of the points (x_1,y_1), (x_2,y_2), $\ldots$ The values of λ are not important from a mathematical point of view, but they have a significant economic interpretation. This method may fail to give the correct answer if there are points belonging to the constraint where $g'_x(x,y)$ and $g'_y(x,y)$ both vanish, but in the cases of our interest this never happens.

4. If the constraint is represented by a bounded set, the problem has solutions, on the base of the theorem of Weierstrass, so we need only to compute the values of the function f at all the points previously obtained: the greatest value is the global maximum, the lowest the global minimum. If the constraint is represented by an unbounded set, the problem is usually two difficult and we'll not be interested in it.

An example will clarify the method.

Example 3. *Maximize/minimize the function*

$$f(x,y)=xy,$$

with the constraint $g(x,y)=x^2+y^2+xy-1=0$, *which is represented by an ellipse of the cartesian plane.*

Solution. Observe that in this case it is not easy to use the equation of the constraint in order to obtain a one variable function from the function f. The constraint is represented by a bounded curve (an ellipse, as said by the text of the problem). So we *must* use the Lagrangian multiplier method. We have

$$\mathcal{L}(x,y,\lambda)=xy-\lambda(x^2+y^2+xy-1).$$

The system of equations to be solved is

$$\begin{cases} y-2\lambda x-\lambda y=0 \\ x-2\lambda y-\lambda x=0 \\ x^2+y^2+xy-1=0 \end{cases} \quad\Rightarrow\quad \begin{cases} y(1-\lambda)=2\lambda x \\ x(1-\lambda)=2\lambda y \\ x^2+y^2+xy-1=0 \end{cases}$$

From the first two equations dividing member to member (it is easy to see that $\lambda=1$, $\lambda=0$, $(x,y)=(0,0)$ can not be solutions of the system) we obtain

$$\frac{y}{x}=\frac{x}{y} \quad\Rightarrow\quad x^2=y^2 \quad\Rightarrow\quad x=\pm y.$$

So the system can be splitted into the systems

$$\begin{cases} x=y \\ x^2+y^2+xy-1=0 \\ y(1-\lambda)=2\lambda x \end{cases} \quad \text{and} \quad \begin{cases} x=-y \\ x^2+y^2+xy-1=0 \\ y(1-\lambda)=2\lambda x \end{cases}.$$

We are only interested in the values of x and y. We obtain the following points:

$$\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right),\quad \left(-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right),\quad (1,-1),\quad (-1,1).$$

As regards the values of f we obtain:

$$f\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)=f\left(-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right)=\frac{1}{3},\quad f(1,-1)=f(-1,1)=-1.$$

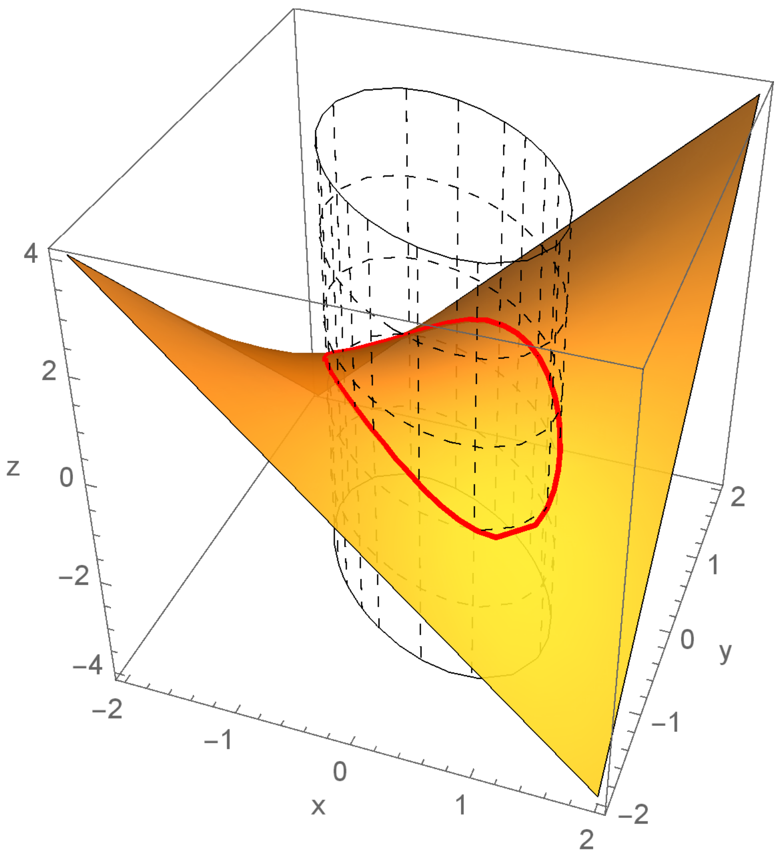
So the global maximum is $1/3$ and the global minimum is -1.

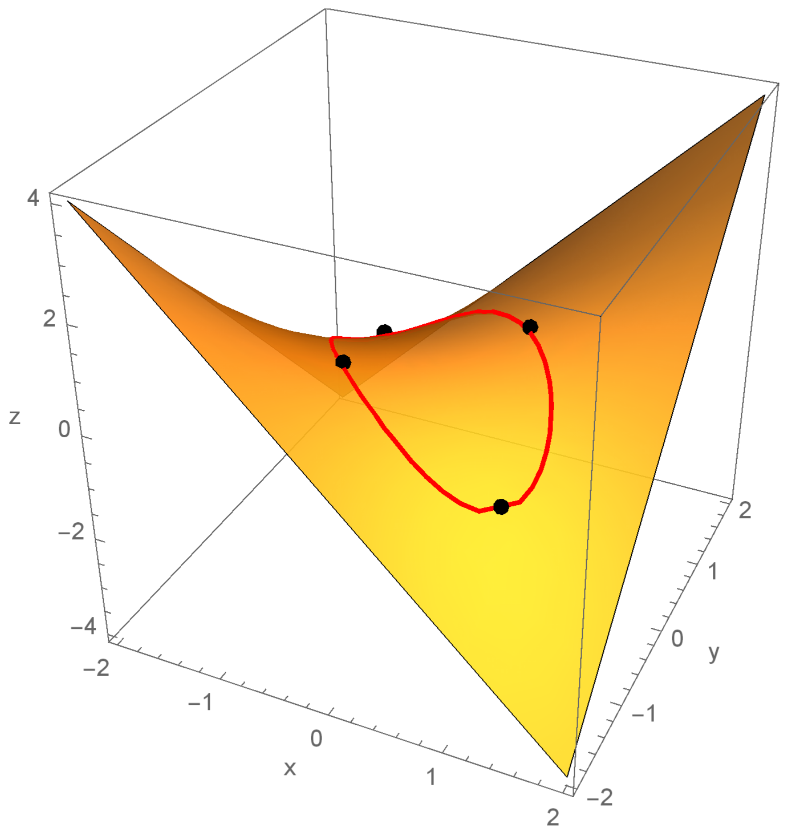
If needed, using the equation

$$y(1-\lambda)=2\lambda x$$

we can also find the values of λ that satisfy the problem. □

Only for the sake of completeness we plot the corresponding graphs, that are very similar to those of example 1.


4
2
z
0
−2
−4
−2
−1
0
x
1
2
2
1
0
y
−1
−2


4
2
z
0
−2
−4
−2
−1
0
x
1
2
2
1
0
y
−1
−2

For the sake of completeness let's consider again the system of equations to be solved in example 3:

$$\begin{cases} y-2\lambda x-\lambda y=0 \\ x-2\lambda y-\lambda x=0 \\ x^2+y^2+xy-1=0 \end{cases} .$$

A more usual strategy to solve the system is "by substitution": find one of the variables using one of the equations and put the obtained expression in the other equations; then proceed in a similar way.

From the first equation we obtain:

$$2\lambda x = y-\lambda y.$$

If λ is 0 we obtain $y=0$ and, from the second equation, $x=0$. But $x=0$, $y=0$ is not a solution of he third equation, so we can suppose $\lambda\neq 0$ and we obtain:

$$x=\frac{y-\lambda y}{2\lambda}.$$

Let's put this value in the second equation; after simplifying we obtain

$$-3\lambda^2 y-2\lambda y+y=0 \quad\Rightarrow\quad -y(3\lambda^2+2\lambda-1)=0 \quad\Rightarrow\quad y=0 \quad\vee\quad 3\lambda^2+2\lambda-1=0.$$

The first possibility is unacceptable (as already proved before) and from the second one we obtain

$$\lambda=-1 \quad\vee\quad \lambda=1/3.$$

From

$$x=\frac{y-\lambda y}{2\lambda}$$

we obtain

$$x=y \quad\vee\quad x=-y\,.$$

Using the third equation we find again the same solutions as before.

There are also other strategies to solve the same system. For example you can substitute the first two equations with their sum or their subtraction. You obtain this new system, after simplifying:

$$\begin{cases} (x+y)(1-3\lambda)=0 \\ (y-x)(1+\lambda)=0 \\ x^2+y^2+xy-1=0 \end{cases} .$$

Proceeding with calculations you obtain again the same solutions (as obvious!).

In general the solution of this system is the more complicated part of the Lagrangian multipliers method.