Mathematics

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Brief summary for second partial - Sample Exercises

1 Two variables functions

Here and in the following the functions are continuous and differentiable as many times as needed.

The main problems considered in our course are the following, for some of which examples are proposed: all these examples have been solved in class. You can also check all the examples proposed in the exercises sheets or the mockups you can find in the usual web site.

1.1 Finding the domain

Usually straight lines, circles, parabolas or simple one variable functions are involved.

1. $f(x,y)=\ln(x^2+y^2-1)$
2. $f(x,y)=\ln(1-x^2-y^2)$
3. $f(x,y)=\sqrt{x^2+y^2-1}$
4. $f(x,y)=\sqrt{1-x^2-y^2}$
5. $f(x,y)=\dfrac{2x^2-3y}{x+y}$
6. $f(x,y)=\sqrt{y-x^2}$
7. $f(x,y)=\ln(y-e^x)$
8. $f(x,y)=\ln(2x+y-1)$

1.2 Studying and plotting level curves

Usually straight lines, circles, parabolas or simple one variable functions are involved.

1. $f(x,y)=\sqrt{x^2+y^2-1}$, curves at levels -1, 0, 1.
2. $f(x,y)=\sqrt{1-x^2-y^2}$, curves at levels -1, 0, 1.
3. $f(x,y)=\dfrac{2x^2-3y}{x+y}$, curves at levels -1, 0, 1.
4. $f(x,y)=\sqrt{y-x^2}$, curves at levels -1, 0, 1.

1.3 Local optimization problems in open subsets of the plane

As far as our course is concerned, *open* means *boundary excluded*, and often means the entire plane. In this case you must use the first order test, that is find the solutions of the system

$$\begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases}$$

and then check the Hessian (second order test).

Pay attention to the fact the points must belong to the interior of the domain. For example if you consider the function $f(x,y) = \ln(x^2 + y^2 - 1)$, the point $(0,0)$ that, at first sight, solves the system of the first order conditions, is not in the domain.

You can try to find local maxima and minima for the functions of subsection ??, except example 5.

1.4 Optimization problems with constraints

As far as our course is concerned the constraints are always curves in the plane (usually straight lines, circles, parabolas or graphs of simple one variable functions).

Two cases have been considered.

1. If it is possible to use the equation of the constraint in order to obtain a single variable function from the given two variables function, then you can use the usual strategy for one variable functions. In this case both local and global optimization problems can be considered, as the limits involved, if there are any, are one variable limits. Pay attention to the restrictions, if there are any, for the residual variable. For example if we need to optimize the function $f(x,y) = x^2 - 2x + y - 1$ on the constraint $y = x^2$, the single variable function obtained by substituting y with x^2 in $f(x,y)$ is $g(x) = 2x^2 - 2x - 1$ with no restriction for x. If we need to optimize the same function on the segment whose boundaries are $A = (1,1)$ and $B = (3,3)$, then the equation of the segment is $y = x$, with $1 \le x \le 3$, the single variable function obtained as usual is $h(x) = 2x^2 - x - 1$, but now x has the restriction $1 \le x \le 3$. We call this strategy *elementary strategy*, and it is by far the simplest one.
2. If it is not possible, or not simple, to use the equation of the constraint in order to obtain a single variable function from the given two variables function, then you must use the Lagrangian multiplier method. In this case:
   a) if the constraint if *closed* and *bounded*, and the question concerns *global optimization* (as is normally the case) you can limit yourself to use the first order check and find the global maximum and minimum by direct comparison of the values of the function in the points that solve the system with first order derivatives;
   b) if the constraint is not bounded we have only considered the possibility of finding *local maxima or minima*, and you must use the bordered Hessian; bordered Hessian can also be used in the previous case if *local maxima or minima* are requested (but we have never considered such a problem).

1.5 Global optimization in closed and bounded subsets of the plane

In this case you must divide the problem in three parts.

1. First, using the strategy outlined in subsection ??, find the possible maximum and minimum points at the *interior* of the given set. Unless explicitly required there is non need to check the Hessian.
2. Second, using the strategy outlined in subsection ??, find the possible maximum and minimum points on the boundary.

3. Then compare the values of the given function at all points previously obtained: the greatest one will be the global maximum, the lowest one the global minimum.

Example 1. *Given the function*

$$f(x,y)=\frac{2x^2-3y}{x+y}$$

find the global maximum and/or minimum on the constraint $x+y+1=0$. (*Use strategy* 1.4.1)

Example 2. *Given the function*

$$f(x,y)=(x-1)^2+y^2$$

find the local maxima and minima using Lagrangian multiplier method on the constraint $x^2-y^2=1$.

Example 3. *Given the function*

$$f(x,y)=(x-1)^2+y^2$$

find the global maximum and/or minimum on the constraint $x=y^2$. (*Use strategy* 1.4.1)

Example 4. *Given the function*

$$f(x,y)=(x-1)^2+y^2$$

find the global maximum and/or minimum on the set given by $x^2+y^2\leq 4$. (*Use strategy* 1.5)

2 Linear algebra

2.1 Preliminaries

1. Matrices and matrix operations (addition, multiplication by a number, product).
2. Determinants (only for *square matrices*; the number obtained is an arbitrary real number).
3. Matrix inversion (only for square matrices with determinant not zero: *non singular* matrices).
4. Rank (for all matrices, even rectangular matrices; the number obtained is a positive integer number, that is 1, 2, 3, ..., except for the null matrix whose rank is 0).

2.2 Linear systems

The first thing to remember is that a linear system can only have

1. one solution (sometimes we write ∞^0 solutions: in this case we agree that this special notation, used *only in this context*, implies that $\infty^0=1$);
2. infinitely many solutions (usually we write ∞^1, ∞^2, ..., depending on the number of unknowns that remain undetermined; when there is only one solution no unknown remains undetermined and this is the origin of the symbol of the previous item);
3. no solution at all.

Systems that have solutions (one or infinitely many) are called *consistent*, system with no solution are called *inconsistent*.

Linear systems can be explicitly written

1. in a traditional form, using equations, unknowns and a connecting brace on the left;
2. using the *coefficients matrix*, the *constants vector* and the *unknowns vector* in the form $A\vec{x}=\vec{b}$;
3. using only the *augmented matrix* $A|b$.

2.3 Non singular square systems

These are systems where the number of equations and the number of unknowns are exactly equal and where $\det(A) \neq 0$.

Such systems can be solved using

1. the inverse matrix strategy: $\vec{x} = A^{-1}\vec{b}$;
2. Cramer's rule.

2.4 General systems

These are rectangular systems or square systems with $\det(A) = 0$. They can be solved using the Rouché-Capelli theorem. If the system is consistent you must have found a minor of the coefficients matrix different from zero. Now you can procede as follows:

1. discard all equations that are non involved in the minor;
2. put all the unknowns whose columns are not involved in that minor at the second member (remember to change the signs!); there are exactly $n-k$ unknowns to be taken to the second member, where k the common rank of the coefficients matrix and the augmented matrix and n the number of unknowns;
3. the remaining system is a square non singular system that can be solved as already said. You obtain ∞^{n-k} solutions, with $n-k$ possibly zero (and in this case you have one only solution!).

Observe that *homogeneous systems*, that is system with all constants equal to 0, are surely consistent and they always have, at least, the solution where the value of all the unknowns is 0. This is a very important consideration for the subject of linear dependence and independence.

2.5 Parametric system

These are systems where the entries of the augmented matrix depend on some parameter (one in our problems, but there can be many in applications!!).

There is nothing special in this situation: only pay attention to the fact all checks you must do in order to solve the system (determinants, ranks) may vary with the value of the parameter, so you need to consider different possibilities, for the different values of the parameter(s). In our problems you have normally two or three special values that require considerations, while in all other cases the behaviour is similar.

2.6 An important application of linear algebra

The application we have considered is the problem of checking if the vectors of a given set are *dependent* or *independent*.

Given r vectors, $\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_r$, with n rows (we shall say vectors of $\mathbb{R}^n$), the simplest strategy to check if they are dependent or independent and to check which of them can, if they are dependent, be written as a linear combination of the others is the following.

1. Write the linear system of n equations in r unknowns $c_1, c_2, c_3, \dots, c_r$ (that are called the coefficients of the combination):

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \cdots + c_r\vec{v}_r = \vec{0}.$$

2. Solve the system, observing that it is an homogeneous system, so it is certainly consistent.
3. If the system has one only solution (the solution where all unknowns are 0) the vectors are *independent*.
4. If the system has infinitely many solutions then the vectors are *dependent*.

In the last case (dependent vectors), all vectors corresponding to non-zero coefficients can be written as a linear combination of the others, all the vectors corresponding to zero coefficients cannot be written as a linear combination of the others.

Example 5. *Consider the vectors*

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ 2 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix},$$

Check if they are dependent or independent and, if they are dependent which of them can be written as a linear combination of the others. When possible write explicitly the linear combination.

Solution. For the sake of simplicity we call the coefficients of the combination x, y, z, t:

$$x\vec{v}_1 + y\vec{v}_2 + z\vec{v}_3 + t\vec{v}_4 = \vec{0}.$$

The coefficients matrix of this system (no need to write the augmented matrix, because it only has an all zeros column in addition) is obtained simply writing the given vectors as columns:

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

This matrix has rank 3 as its determinant is zero but, for instance, the submatrix obtained at the intersection of rows 2, 3, 4 and columns 2, 3, 4 has determinant −2.

The reduced system (discard the first equation and take the x to the second member) is

$$\begin{cases} y - z - t = 0 \\ y + z = -x \\ 2z = -x \end{cases}.$$

The $\left(\infty^{4-3} = \infty^1\right)$ solutions are

$$\left(x, -\frac{x}{2}, -\frac{x}{2}, 0\right),$$

for all values of x.

We can rewrite the combination as follows:

$$x\vec{v}_1 - \frac{x}{2}\vec{v}_2 - \frac{x}{2}\vec{v}_3 + 0v_4 = \vec{0}.$$

So we obtain

$$\vec{v}_1 = \frac{1}{2}\vec{v}_2 + \frac{1}{2}\vec{v}_3 \left(+0\vec{v}_4\right), \quad \vec{v}_2 = 2\vec{v}_1 - \vec{v}_3 \left(+0\vec{v}_4\right), \quad \vec{v}_3 = 2\vec{v}_1 - \vec{v}_2 \left(+0\vec{v}_4\right),$$

while $\vec{v}_4$ cannot be written as a linear combination of the others. □